Whether to Hire A(nother) Superstar*

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January 16, 2020

Abstract

Empirical results show that the introduction of a superstar – an opponent with relatively high ability – into a contest may have opposite effects: sometimes it increases other participants' performance while other times decreases it. We present a cohesive model capturing both effects and characterize the conditions under which the entry of a superstar, and more generally a high-ability participant, encourages/discourages other participants and increases/decreases their performance. In addition, we find that if the incremental difference in prize values grows sufficiently fast in prize ranking, we cannot keep recruiting high-ability participants without suffering from the discouraging effect.

JEL classification: D44, J31, D72

Keywords: contest, entry, superstar

*The author would like to thank Vijay Krishna for his comments on early drafts, Ivan Balbuzanov, Simon Loertscher, Claudio Mezzetti, Steven Williams and seminar participants in Australian Economic Theory Workshop, Lingnan Workshop on Microeconomic Theory and Experiment, Monash University, SAET, University of Melbourne, UNSW, University of Queensland for comments and discussion, and Jonathan Lim for research assistance. Financial support from the University of Melbourne is also greatly appreciated.

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1 Introduction

In 1930, the cycling community was startled to learn that Giro d'Italia, one of the most prestigious bicycle races, would run without Alfredo Binda, who had won the race in three previous years. In fact, the race organizers paid the top cyclist 22,500 lire, an amount equal to the first place prize, to *miss* the race, because they believed his dominant riding was suppressing the competition.[1] Similar discouraging effects of superstars arise in other sports. For instance, Brown (2011) studies professional golf tournaments from 1999-2006 and finds that in the presence of Tiger Woods, the dominating golfer in that period, the other golfers performed worse by 0.8 strokes, where one stroke frequently makes the difference between the champion and second place.

However, rivalry may encourage competitors to exert more effort. Indeed, Mike Powell attributes his long jump world record to his rivalry with Carl Lewis, a nine time Olympic gold medalist: "*He (Lewis) motivated me and drove me to do big things ... I had to break an unbreakable world record just to beat this guy.*"[2] Similarly, in the presence of Usain Bolt, a dominant sprinter during 2008-2017, other participants were more likely to break their personal records in 100-meter races (Hill 2014). In golf, when Tiger Woods won the 2019 Masters Tournament after a long period plagued with injuries, the 2018 champion Patrick Reed said: "*Tiger is back, and we're going to have to step up our games.*"[3]

When does the entry of a superstar, and more generally a participant with relatively higher ability, encourage other participants' performance, and when does the entry discourage their performance? What are the circumstances in which the discouraging effect imposes barriers to improve participants' ability? These questions are important to not only organizers of sports tournaments but also to schools awarding scholarships according to students' grade rankings, and firms using internal rank-competition to incentivize employees. Those firms include Amazon, General Electric, General Motors, Hewlett-Packard, IBM, Johnson&Johnson, Microsoft, Motorola and Yahoo.

In this paper, we provide a cohesive framework to capture both the encouraging and discouraging effects. We illustrate our setup in an example below. Consider a contest with four participants: two high-ability ones whose marginal cost of performance is 1, and two low-ability ones whose marginal cost of performance is 3. In this paper, we study participants of two ability types. Information is complete so each player's ability is commonly known. The contest has a sequence of four prizes, e.g., \$0, \$1, \$5, \$13. In this paper, we study prize sequences with a constant second order difference, e.g., $(\$5 - \$1) - (\$1 - \$0) = (\$13 - \$5) - (\$5 - \$1) = \$3$, and use the second order difference to measure the convexity of prize sequences. The players choose their performance simultaneously. The player with the highest performance wins the highest prize, the player with the second highest performance wins the second highest prize, and so on.[4]

[1] See McGann and McGann (2011).
[2] See Ganguly (2011).
[3] See Brown (2019).
[4] In case of a tie, the relevant prizes are allocated among tying participants with equal probabilities.

In general, the combination of asymmetric abilities and heterogenous prizes makes it challenging to characterize the Nash equilibrium. However, in our model, we can use techniques from combinatorics to characterize the unique equilibrium strategies as solutions of quadratic equations. As a result, we obtain a tractable model of contests, in which the equilibrium performance depends on three characteristics: 1) the prize structure, 2) ability composition (proportion of high-ability players), and 3) the difference between the two ability types.

In order to measure the effect of a high-ability player's presence, we adopt Brown's (2011) approach to identify the Tiger Woods' effect. Specifically, consider a high-ability player entering the contest who replaces a low-ability player. As a result, the size of the contest as well as its prizes remain the same. For instance, if a high-ability player enters a contest with two high-ability players and two low-ability ones, there are three players with high ability and one with low ability. Moreover, three players remain the same before and after the entry. We compare the performances of those players before and after the entry, and ask in which contests, characterized by their prize structure, fraction of high-ability players, and the difference between ability types, does the entry discourage their performance and in which does the entry encourage their performance.

We find that the entry effects vary with the convexity of the prize sequence, which is measured by the second order difference. Specifically, if the prize sequence is not very convex (with a small second order difference, e.g., \$0, \$1, \$2, \$3),

(a) the entry of a(nother) high-ability player encourages the existing high-ability players' performance but discourages the existing low-ability players;

(b) the entry of a(nother) high-ability player discourages the existing players' total performance if and only if the proportion of high-ability players is low.

In contrast, if the prize sequence is convex enough (with a large second order difference, e.g., \$0, \$1, \$5, \$13), the above results surprisingly change to

(a′) the entry of a(nother) high-ability player *discourages* every existing player's performance;

(b′) the entry of a(nother) high-ability player discourages the existing players' total performance if and only if player abilities are more homogeneous (high fraction for either ability type).

We apply the above results to study barriers to improve participants/employees' ability. Specifically, can we improve the participants' ability without incurring the discouraging effect? If the prize sequence is sufficiently convex, improving an employee's ability may first encourage the other employees when the high-ability proportion is low, but eventually discourages the others as the high-ability proportion becomes sufficiently high. Hence, to avoid the discouraging effect, the prize sequence eventually needs to be adjusted to be less convex.

Notice that there are two main differences between the two sets of results if the prize sequence becomes more convex: first, between (a) and (a′), the entry's effect on the high-ability players is reversed. Second, between (b) and (b′), given sufficiently high proportion of high-ability, the entry's effect on the existing players' total performance is also reversed, from encouraging to discouraging.

We first explain the difference between results (a) and (a$'$). Namely, why, as the prize sequence becomes convex enough, is the encouraging effect on the high-ability players reversed to the discouraging effect? To understand the intuition, consider the above contest between two high-ability players and two low-ability ones, and let the prizes be $\$0, \$1, \$2, \$3$. In addition, assume that the marginal performance cost of the high-ability players is almost zero, which is an extreme case of our model but makes the following explanation simpler. Due to the almost costless performance, a high-ability player can ensure the second highest prize \$2 by beating the two low-ability players. After the entry, she can ensure only the third highest prize \$1 because there are fewer low-ability players. Thus, the first effect of the entry is lower guaranteed winnings for the high-ability players. Since this effect is from low-ability players towards the high-ability ones, we call it the *incentive-from-bottom* effect. Given everything else, this effect puts an upward pressure on his performance.

The second effect of the entry is reducing the expected winnings of a high-ability player, because there are more high-ability players to share the top prizes. The expected winnings of a high-ability player is the average of the top two prizes $(\$3 + \$2)/2$ before the entry, and reduces to the average of the top three prizes $(\$3 + \$2 + \$1)/3$ afterwards. Since this effect is from the top prizes, we call it *disincentive-from-top* effect. Given everything else, this effect puts a downward pressure on his performance.

Both effects vary with the convexity of the prize sequence, but in different ways. For instance, given the sequence of $\$0, \$1, \$2, \$3$, the disincentive-from-top effect, measured by the change in the average prize values, is $\frac{\$3+\$2}{2} - \frac{\$3+\$2+\$1}{3}$. If the prize sequence changes to a more convex one of $\$0, \$1, \$5, \$13$, the disincentive-from-top effect becomes $\frac{\$13+\$5}{2} - \frac{\$13+\$5+\$1}{3}$, which is increased by $\frac{13}{6}$. In contrast, with the sequence of $\$0, \$1, \$2, \$3$, the incentive-from-bottom effect, measured by the difference between the second and third highest prizes, is $\$2 - \$1$. Changing the prize sequence to $\$0, \$1, \$5, \$13$, the incentive-from-bottom effect becomes $\$5 - \$1$, which is increased by 2. Notice that, as the prize sequence becomes more convex, the disincentive-from-top effect grows more than the incentive-from-bottom effect. Thus, with a sufficiently convex prize sequence, the disincentive-from-top effect dominates the incentive-from-bottom effect, and therefore the entry discourages the high-ability players.

In the general case, when the high-ability players' performing cost is not necessarily close to zero, the payoff effect depends on not only the guaranteed prize but also the cost of the necessary performance to win the prize. We show in the general case, as the prize sequence becomes more convex, the disincentive-from-top effect still grows faster than the incentive-from-bottom effect.

Next, we explain the difference between results (b) and (b$'$): given sufficiently high proportion of high-ability, the entry's effect on the others' total performance is also reversed, from encouraging to discouraging. This difference is simply implied by the first difference explained above. To see this, notice that if the contest has a high fraction of high-ability players, the first main difference implies that the entry's effect on the high-ability players is reversed from

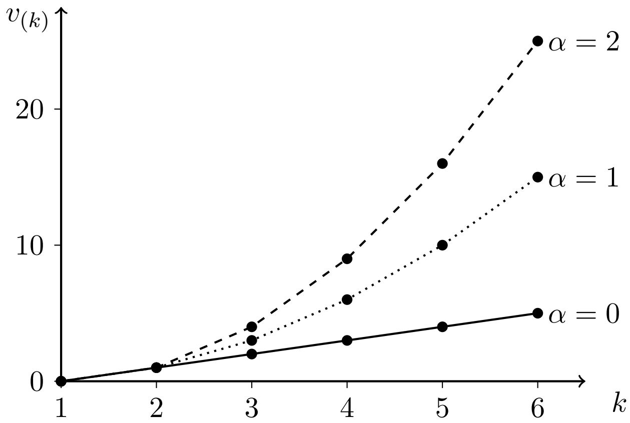

Figure 1: Prize Sequences

encouraging to discouraging. Since most of the other players have high ability, the entry's effect on their total performance is also reversed from encouraging to discouraging.

The remainder of the paper is organized as follows. Section 2 introduces the model of contest and Section 3 characterizes the equilibrium. Section 4 identifies the circumstances for the encouraging and discouraging effects to arise. Section 5 analyzes the barriers against improving participants' ability, and Section 6 discusses the entry's effects on total performance. Section 7 discusses related literature and Section 8 concludes.

2 Model

Consider a contest in which $n \geq 3$ players compete for n prizes with heterogeneous monetary values: $v_{(n)} > v_{(n-1)} > ... > v_{(1)} = 0$. We can normalize the prize values so that $v_{(2)} = 1$. Moreover, the prizes have constant second order differences: $(v_{(k+1)} - v_{(k)}) - (v_{(k)} - v_{(k-1)}) = \alpha \geq 0$ for $k = 2, ..., n-1$. As a result, the prize sequence is characterized by its convexity parameter α. Indeed, in a contest among n players, the prize values are $v_{(k)} = C_1^{k-1} + \alpha C_2^{k-1}$ for $k = 1, ..., n$, where $C_k^m = \frac{n!}{k!(n-k)!}$ is the binomial coefficient and $C_k^m = 0$ if $m < k$. Figure 1 illustrates prize sequences with differences in convexity. Since $\alpha \geq 0$, the prize sequence is always (weakly) convex.[5] As in the figure, if $\alpha = 0$, the prize sequence is linear. As α increases, the prize sequence becomes more convex in the sense that the differences between higher-value prizes grow bigger relative to those between lower-value prizes.

In the contest, the players choose costly performance to compete.[6] The players have constant marginal costs of performance. The marginal costs can be c_H or c_L with $0 < c_L < c_H$. We refer to the players with marginal cost c_H as H-cost players, and those with c_L as L-cost ones. The H-cost players are less productive because their marginal costs are higher. Denote the number of L-cost players as n_L. Then, we can characterize the composition of participants by n_L/n, the proportion of L-cost players. As n_L/n increases from 0 to 1, the players' average marginal

[5]Convex prize sequences are prevalent in sports and organizations.
[6]In this paper, we do not distinguish performance and effort.

cost decreases, therefore they are more productive.

The ratio $c_L/c_H \in (0,1)$ measures the asymmetry between the two cost types. We assume $c_L/c_H < 1/2$, and it corresponds to the case in which the cost types are significantly different.[7] The three parameters discussed above are important for the rest of the paper: α the convexity of the prize sequence, n_L/n the proportion of L-cost players, and c_L/c_H the asymmetry between the cost types.

Therefore, with its size fixed, a contest can be described by three characteristics: the convexity of prize sequence measured by α, the composition of types measured by n_L/n and the asymmetry between types measured by c_L/c_H.

The game is of complete information, so each player's cost is commonly known. In the contest, each participant i chooses his performance $s_i \geq 0$ simultaneously. Note that there is no noise in their performance.[8] The participant with the highest performance receives the highest prize $v_{(n)}$; the second highest performance receives the second highest prize $v_{(n-1)}$; and so on. In the case of a tie, the involved prizes are split evenly among the tying participants. All the players are risk neutral. If a t-cost player wins prize $v_{(k)}$ with performance s_i, his payoff is $v_{(k)} - c_t s_i$, where $t = H$ or L.

We use a c.d.f. $G_i : [0, +\infty) \to [0,1]$ to represent player i's mixed strategy. The support of G_i is the smallest closed set to which G_i assigns probability 1. If its support is a singleton, a mixed strategy reduces to a pure strategy. A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players. Throughout the paper, we consider type-symmetric Nash equilibrium, where players of the same cost use the same strategy. We focus on type-symmetric equilibrium purely for simpler analysis.[9]

3 Equilibrium Characterization

In this section, we characterize the unique equilibrium in closed form. With heterogeneous prize values and asymmetric player abilities, equilibria generally involve complicated mixed strategies without closed-form characterization (e.g. Xiao 2016). However, using techniques in combinatorics, we can simplify the characterization to solving a single quadratic equation. Those techniques are useful because binomial coefficients frequently arise in equilibrium analysis through two channels: First, as a result of constant second order differences, the prize values depend on binomial coefficients: $v_{(k)} = C_1^{k-1} + \alpha C_2^{k-1}$. Second, a player's winning probabilities also depend on binomial coefficients. For example, facing four H-cost opponents, a player choosing performance s wins the third highest prize with probability $C_2^4 G_H^2(s)(1 - G_H(s))^2$.

[7] This assumption also simplifies equilibrium charaterization because it ensures each player mixes over an interval of performance levels.

[8] Our model is different from tournament models (e.g. Rosen 1981), which study moral hazard arising in the presence of noisy performance. In our model, information is complete therefore moral hazard does not arise.

[9] If idiosyncratic shocks are added to the marginal costs, so player i's marginal cost becomes $c_i = c_t + \varepsilon_i$ and no two players have the same marginal cost. Then, there is a unique Nash equilibrium (Xiao 2016). Moreover, the type-symmetric equilibrium is the limit of the unique Nash equilibrium as the shocks converge to zero.

Appendix A contains a complete list of results from combinatorics that we use in this paper.

Depending on the composition of player types, the equilibrium may be one of four types. Each type is named after its characteristic equilibrium property and is discussed separately in Sections 3.1-3.4 below.

3.1 Equilibrium with Symmetry

First, consider a contest with only one type of players. This is a contest with symmetric players and the equilibrium is well understood (see for instance Barut and Kovenock 1998). Each player's equilibrium payoff is zero, and this contest has a unique equilibrium. Moreover, the equilibrium has symmetric mixed strategies with support $[0, v_{(n)}/c_t]$. Example 1 illustrates such an equilibrium.

Example 1 (Equilibrium with Symmetry) *Consider a contest with four H-cost players with $c_H = 3$. The prizes are $v_{(4)} = 9$, $v_{(3)} = 4$, $v_{(2)} = 1$ and $v_{(1)} = 0$, which has a constant second order difference $\alpha = 2$. The unique equilibrium is illustrated in Figure 2. Notice that the all the players mix over the interval $[0, 3]$ according to the same mixed strategy G_H.*

We explain below how to derive the equilibrium strategy in Example 1. Denote the symmetric strategy as G_H. If all others use strategy G_H, a player should receive his equilibrium payoff by choosing any s in the support of G_H:

$$v_{(4)}G_H^3(s) + v_{(3)}C_2^3 G_H^2(s)(1 - G_H(s)) + v_{(2)}C_1^3 G_H(s)(1 - G_H(s))^2 + v_{(1)}(1 - G_H(s))^3 - c_H s = 0$$

For cleaner exposition, we often omit the argument of $G_t(s)$ below. Collecting terms w.r.t. G_H, we can rewrite the above equation as

$$\Delta_3 G_H^3 + \Delta_2 C_2^3 G_H^2 + \Delta_1 C_1^3 G_H + v_{(1)} - c_H s = 0 \tag{1}$$

where $\Delta_1 = v_{(2)} - v_{(1)}$, $\Delta_2 = (v_{(3)} - v_{(2)}) - (v_{(2)} - v_{(1)})$, $\Delta_3 = [(v_{(4)} - v_{(3)}) - (v_{(3)} - v_{(2)})] - [(v_{(3)} - v_{(2)}) - (v_{(2)} - v_{(1)})]$. Recall that $v_{(1)} = 0$ and $v_{(2)} = 1$, so $\Delta_1 = 1$. Moreover, the constant second order differences imply $\Delta_2 = \alpha$ and $\Delta_3 = 0$. Therefore, we can further simplify the equation to

$$\alpha C_2^3 G_H^2 + C_1^3 G_H - c_H s = 0 \tag{2}$$

which has a unique solution in $[0, 1]$ and it is

$$G_H(s) = \frac{-C_1^3 + \sqrt{(C_1^3)^2 + 4\alpha C_2^3 c_H s}}{2\alpha C_2^3}$$

More generally, for $t = H$ or L, if a contest is among n identical t-cost players, the counter-

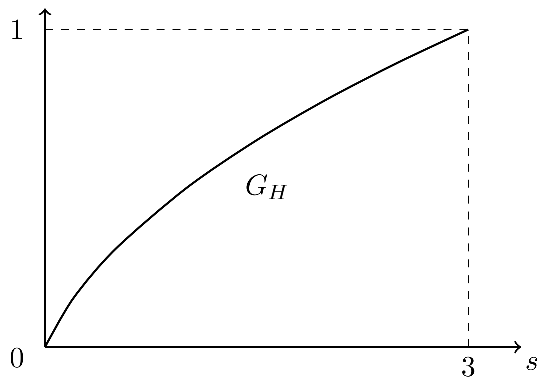

Figure 2: Equilibrium with Symmetry

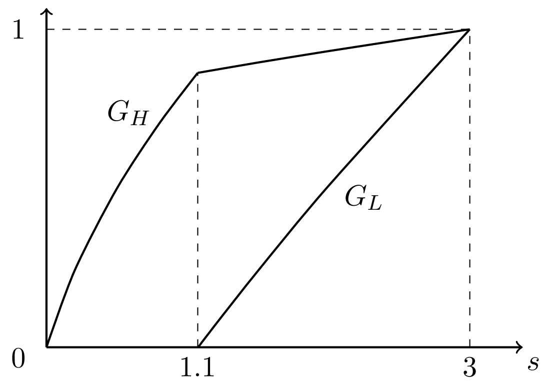

Figure 3: Equilibrium with Embedment

part of (2) is $\alpha C_2^{n-1} G_t^2 + C_1^{n-1} G_t - c_t s = 0$, whose unique solution in $[0, 1]$ is

$$G_t(s) = \frac{-C_1^{n-1} + \sqrt{(C_1^{n-1})^2 + 4\alpha C_2^{n-1} c_t s}}{2\alpha C_2^{n-1}} \tag{3}$$

The following result characterizes the equilibrium with symmetry in an n-player contest.

Proposition 1 (Equilibrium with Symmetry) *If a contest has only t-cost players with $t = H$ or L, there is a unique equilibrium. In the equilibrium, all the players have the same payoff $u_t = 0$ and use a symmetric mixed strategy $G_t(s)$ given in (3) for $s \in [0, v_{(n)}/c_t]$.*

3.2 Equilibrium with Embedment

Next, consider a contest in which all players except one have the low cost, so $n_L = 1$ and $n_H = n - 1$. In the equilibrium, the L-cost player mixes over a higher interval $[\underline{s}_L, v_{(n)}/c_H]$ and the H-cost players mix over a lower interval $[0, v_{(n)}/c_H]$. The L-cost's interval is a subset of the H-cost's, and they share the same upper boundary. Example 2 illustrates such an equilibrium.

Example 2 (Equilibrium with Embedment) *Consider a contest with three H-cost players with marginal cost $c_H = 3$ and one L-cost player with $c_L = 1$. The prizes are the same as in Example 1. The unique equilibrium is illustrated in Figure 3. Notice that the support of G_L is $[1.1, 3]$ and that of G_H is $[0, 3]$. The equilibrium payoffs are $u_H = 0$ for the H-cost players and $u_L = 6$ for the L-cost player.*

To understand the payoffs, notice that the H-cost players have payoff $u_H = 0$. The highest performance in G_H's support is $v_{(n)}/c_H$, the same as in Example 1. Any performance above $v_{(n)}/c_H$ never earns positive payoff for an H-cost player. It turns out that G_H and G_L have to share the same upper boundary in an equilibrium. Therefore, if the L-cost player chooses the upper boundary, he wins $v_{(n)}$ with probability 1 and his payoff is $u_L = v_{(n)} - c_L v_{(n)}/c_H$.

To illustrate the main idea, we derive below the equilibrium strategies in Example 2 using u_L and u_H. The analysis for the general case is in the appendix. First, consider interval $[0, 1.1]$. Given others' equilibrium strategies, if an H-type player chooses a performance in the interval, he never wins the highest prize because the L-cost player's performance is always higher. Therefore, by choosing $s \in [0, 1.1]$, an H-cost player is competing with the other two H-cost players for the prizes $v_{(3)}$ and $v_{(2)}$. We obtain a condition similar to (2):

$$\alpha C_2^2 G_H^2 + C_1^2 G_H - c_H s = 0$$

where C_1^2 and C_2^2 replace C_1^3 and C_2^3 in (2). This is because each H-cost player compete against only two other H-cost players. The above equation has a unique solution in $[0, 1]$:

$$\hat{G}_H(s) = \begin{cases} \dfrac{-C_1^2 + \sqrt{(C_1^2)^2 + 4\alpha C_2^2 c_H s}}{2\alpha C_2^2} & \text{if } \alpha > 0 \\ c_H s / C_1^2 & \text{if } \alpha = 0 \end{cases} \tag{4}$$

Next, consider the interval $[1.1, 3]$. Given others' equilibrium strategy G_H, the L-cost player receives his equilibrium payoff by choosing a performance in the interval:

$$\alpha C_2^3 G_H^2 + C_1^3 G_H - c_L s = u_L$$

Notice that the expected prize, the first two terms, is the same as in (2) because the opponents are the same: three H-cost players. The above equation has unique solution in $[0, 1]$:

$$\bar{G}_H(s) = \begin{cases} \dfrac{-C_1^3 + \sqrt{(C_1^3)^2 + 4\alpha C_2^3 (c_L s + u_L)}}{2\alpha C_2^3} & \text{if } \alpha > 0 \\ (u_L + c_L s)/C_1^3 & \text{if } \alpha = 0 \end{cases} \tag{5}$$

For $s \in [1.1, 3]$, the counterpart for an H-cost player is

$$\begin{aligned} v_{(4)} \bar{G}_H^2 G_L \quad &+ \quad v_{(3)} [C_1^2 \bar{G}_H (1 - \bar{G}_H) G_L + C_2^2 \bar{G}_H^2 (1 - G_L)] \\ &+ \quad v_{(2)} [C_2^2 (1 - \bar{G}_H)^2 G_L + C_1^2 \bar{G}_H (1 - \bar{G}_H)(1 - G_L)] - c_H s = 0 \end{aligned}$$

or

$$\Delta_3 \bar{G}_H^2 G_L + \Delta_{(2)} (C_1^2 \bar{G}_H G_L + C_2^2 \bar{G}_H^2) + \Delta_1 (C_1^2 \bar{G}_H + G_L) - c_H s = 0$$

Since $\Delta_1 = 1$, $\Delta_2 = \alpha$ and $\Delta_3 = 0$, we can further simplify the equation to

$$\alpha (C_1^2 \bar{G}_H G_L + C_2^2 \bar{G}_H^2) + (C_1^2 \bar{G}_H + G_L) - c_H s = 0$$

which implies

$$G_L = \frac{c_H s - \alpha C_2^2 \bar{G}_H^2 - C_1^2 \bar{G}_H}{\alpha C_1^2 \bar{G}_H + 1} \tag{6}$$

Hence, $G_H(s)$ for $s \in [0, 1.1]$ is given in (4), and $G_H(s)$ for $s \in [1.1, 3]$ is given in (5). Substituting $\bar{G}_H$ given in (5) into (6), we obtain $G_L(s)$ for $s \in [1.1, 3]$.

Following the same argument, we can obtain the counterparts of (4)-(6) in an n-player contest where all but one players are H-cost:

$$\hat{G}_H(s) = \begin{cases} \dfrac{-C_1^{n-2} + \sqrt{(C_1^{n-2})^2 + 4\alpha C_2^{n-2} c_H s}}{2\alpha C_2^{n-2}} & \text{if } \alpha > 0 \\ c_H s / C_1^{n-2} & \text{if } \alpha = 0 \end{cases} \tag{7}$$

$$\bar{G}_H(s) = \begin{cases} \dfrac{-C_1^{n-1} + \sqrt{(C_1^{n-1})^2 + 4\alpha C_2^{n-1}(c_L s + u_L)}}{2\alpha C_2^{n-1}} & \text{if } \alpha > 0 \\ (u_L + c_L s)/C_1^{n-1} & \text{if } \alpha = 0 \end{cases} \tag{8}$$

$$G_L(s) = \frac{c_H s - \alpha C_2^{n-2} \bar{G}_H^2(s) - C_1^{n-2} \bar{G}_H(s)}{\alpha C_1^{n-2} G_H(s) + 1} \tag{9}$$

and $\underline{s}_L$ is the performance level in $(0, v_n/c_H)$ such that $G_L(\underline{s}_L) = 0$.[10] The equilibrium characterization generalizes as follows:

Proposition 2 (Equilibrium with Embedment) *If $n_L = 1$, there is a unique equilibrium, and the equilibrium payoffs are $u_L = v_{(n)}(1 - c_L/c_H)$ for the L-cost player and $u_H = 0$ for the H-cost players.*

In the equilibrium, the L-cost player mixes over $[\underline{s}_L, v_{(n)}/c_H]$ and the H-cost players mix over $[0, v_{(n)}/c_H]$. Strategy $G_H(s)$ for $s \in [0, \underline{s}_L]$ is given in (7) and $G_H(s)$ for $s \in [\underline{s}_L, v_{(n)}/c_H]$ is given in (8). Strategy $G_L(s)$ for $s \in [\underline{s}_L, v_{(n)}/c_H]$ is given in (9).

All the proofs are in the appendix.

3.3 Equilibrium with Separation

If a contest has $n_L \geq 2$ L-cost players and $n_H \geq 2$ H-cost players, the H-cost players mix over a lower interval $[0, \frac{v_{(n_H)}}{c_H}]$ and the L-cost players mix over a higher interval $[\frac{v_{(n_H)}}{c_H}, \frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}]$. Notice that the two intervals share a boundary but do not overlap. As a result, the competition is separated: the L-cost players compete for $v_{(4)}$ and $v_{(3)}$ while the H-cost players compete for $v_{(2)}$ and $v_{(1)}$. Example 3 illustrates such an equilibrium.

Example 3 (Equilibrium with Separation) *Consider a contest with two H-cost players with $c_H = 3$ and two L-cost players with $c_L = 1$. The prizes are the same as in Example 1. The unique equilibrium is illustrated in Figure 4. Notice that the support of G_L is $[1/3, 5]$ and that of G_H is $[0, 1/3]$. The equilibrium payoffs are $u_H = 0$ for the H-cost players and $u_L = 4$ for the L-cost player.*

We derive below the equilibrium strategies in the example. Since the competition is separated, the two H-cost players compete for $v_{(2)}$ as if they are in a two-player contest. Thus, his equilibrium payoff is $u_H = 0$ and equilibrium strategy is $G_H(s) = c_H s / v_{(2)}$. The highest performance in the support of G_H is $v_{(2)}/c_H$, which is also the lowest performance in the support of G_L. At this performance, an L-cost player wins $v_{(3)}$ with certainty, so his payoff is

[10] The closed-form expression of $\underline{s}_L$ is provided in the Appendix below (33).

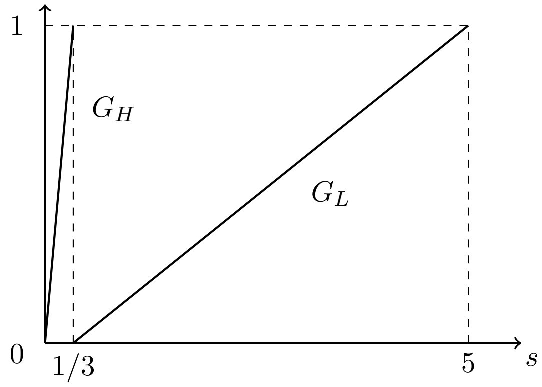

Figure 4: Equilibrium with Separation

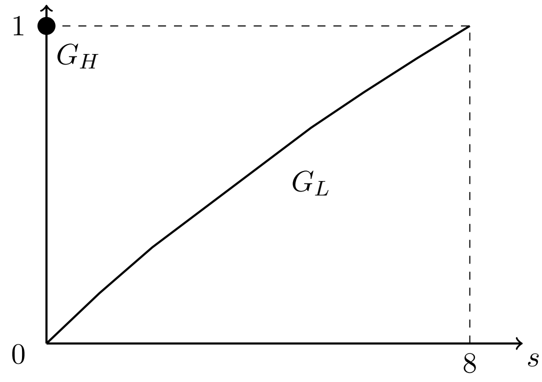

Figure 5: Equilibrium with Nonperformance

$u_L = v_{(3)} - c_L v_{(2)}/c_H$. The two L-cost players compete for $v_{(4)}$ and $v_{(3)}$ as if they are in a two-player contest. Hence, their equilibrium payoffs are $u_L = v_{(3)}$ and equilibrium strategies satisfy $v_{(4)}G_L + v_{(3)}(1 - G_L) - c_L s = u_L$, so $G_L(s) = (u_L + c_L s - v_{(3)})/(v_{(4)} - v_{(3)})$.

More generally, consider a contest with $n_L \geq 2$ L-cost players and $n_H \geq 2$ H-cost players. With separated competition, the H-cost players compete for $v_{(1)}, ..., v_{(n_H)}$ as if they are in a contest with n_H players. Similar to (3), their equilibrium strategy G_H satisfies

$$\alpha C_2^{n_H-1} G_H^2 + C_1^{n_H-1} G_H - c_H s = 0 \tag{10}$$

where $C_2^{n_H-1} = 0$ if $n_H = 2$. The unique solution in $[0, 1]$ is

$$G_H(s) = \begin{cases} \dfrac{-C_1^{n_H-1} + \sqrt{(C_1^{n_H-1})^2 + 4\alpha C_2^{n_H-1} c_H s}}{2\alpha C_2^{n_H-1}} & \text{if } n_H \geq 3 \\ c_H s & \text{if } n_H = 2 \end{cases} \tag{11}$$

The highest performance in the support of G_H is $v_{(n_H)}/c_H$. As above if an L-cost player chooses this performance, he receives $v_{(n_H+1)}$ and obtains his equilibrium payoff $u_L = v_{(n_H+1)} - c_L v_{(n_H)}/c_H$. The n_L L-cost players compete for $v_{(n_H+1)}, ..., v_{(n)}$ as if they are in an n_L player contest. Therefore, their strategy satisfies an analogue of (1):

$$\sum_{m=0}^{n_L-1} \Delta_{n_H+1}^m C_m^{n_L-1} G_L^m - c_L s = u_L$$

where $\Delta_k^0 = v_{(k)}$ for $k = 1, ..., n$ and $\Delta_k^m = \Delta_{k+1}^{m-1} - \Delta_k^{m-1}$ for $k = 1, ..., n - m$. Using techniques from combinatorics (see Appendix A), we can verify that $\Delta_{n_H+1}^m = 0$ for $m \geq 3$, $\Delta_{n_H+1}^2 = \alpha$ and $\Delta_k^1 = C_0^{k-1} + \alpha C_1^{k-1}$. Moreover, we have $u_L = v_{(n_H+1)} - c_L v_{(n_H)}/c_H$, so the above equation can be simplified to

$$\alpha C_2^{n_L-1} G_L^2 + \Delta_{n_H+1}^1 C_1^{n_L-1} G_L + v_{(n_H+1)} - c_L s = v_{(n_H+1)} - c_L v_{(n_H)}/c_H \tag{12}$$

whose unique solution in $[0,1]$ is

$$G_L(s) = \begin{cases} \dfrac{-\Delta^1_{n_H+1} C_1^{n_L-1} + \sqrt{(\Delta^1_{n_H+1} C_1^{n_L-1})^2 + 4\alpha C_2^{n_L-1} c_L (s - v_{(n_H)}/c_H)}}{2\alpha C_2^{n_L-1}} & \text{if } n_L \geq 3 \\ \dfrac{c_L(s - v_{(n-2)}/c_H)}{v_{(n)} - v_{(n-1)}} & \text{if } n_L = 2 \end{cases} \tag{13}$$

The equilibrium characterization generalizes as follows

Proposition 3 (Equilibrium with Separation) *If $2 \leq n_L \leq n-2$, there is a unique equilibrium, and the equilibrium payoffs are $u_H = 0$ for the H-cost players and $u_L = v_{(n_H+1)} - c_L v_{(n_H)}/c_H$ for the L-cost players.*

In the equilibrium, G_H has support $[0, \frac{v_{(n_H)}}{c_H}]$ and G_L has support $[\frac{v_{(n_H)}}{c_H}, \frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}]$. Moreover, $G_H(s)$ is given in (11) and $G_L(s)$ in (13).

3.4 Equilibrium with Nonpeformance

If a contest has $n-1$ L-cost players and one H-cost player, then the H-cost player chooses nonperformance with certainty and the $n-1$ L-cost players compete for the $n-1$ positive prizes. Example 4 illustrates such an equilibrium.

Example 4 (Equilibrium with Nonperformance) *Consider a contest with one H-cost player with $c_H = 3$ and three L-cost players with $c_L = 1$. The prizes are the same as in Example 1. In the equilibrium, the H-cost player chooses nonperformance and the L-cost players' equilibrium strategies are illustrated in Figure 5. The equilibrium payoffs are $u_H = 0$ and $u_L = 1$.*

To understand an L-cost player's payoff, suppose others choose their equilibrium strategies and he chooses performance slightly above 0. Then, he beats the H-cost player while losing to the other L-cost players, so he wins $v_{(2)} = 1$ with almost no cost. Thus, his equilibrium payoff is $u_L = 1$.

Notice that the above equilibrium is an extreme case of *equilibrium with separation* with the support of G_H reduced to a singleton. Therefore, substituting $\Delta^1_2 = 1 + \alpha$, $n_H = 1$ and $u_L = 1$ into (13) we obtain

$$G_L(s) = \begin{cases} \dfrac{-(1+\alpha)C_1^{n-2} + \sqrt{\left[(1+\alpha)C_1^{n-2}\right]^2 + 4\alpha C_2^{n-2} c_L s}}{2\alpha C_2^{n-2}} & \text{if } n_L \geq 3 \\ \dfrac{c_L s}{v_{(3)} - 1} & \text{if } n_L = 2 \end{cases} \tag{14}$$

Thus, the equilibrium characterization is as follows:

Proposition 4 (Equilibrium with Nonperformance) *In a contest with one H-cost player and $n-1$ L-cost players, there is a unique equilibrium and the equilibrium payoffs are $u_H = 0$ for the H-type players and $u_L = 1$ for the L-type players.*

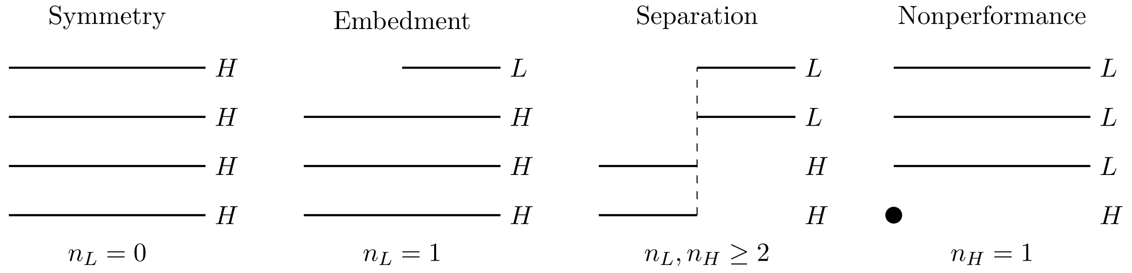

Figure 6: Four Equilibrium Types

In the equilibrium, the H-cost player chooses nonperformance with certainty and the L-cost players' strategies are given in (14).

So far, we have introduced four equilibrium types. Combining Propositions 1-4, we obtain

Corollary 1 *The contest has a unique equilibrium. Moreover, the payoffs and strategies in the equilibrium can be expressed in closed form and they are are given in Propositions 1-4.*

Figure 6 illustrates the strategies' supports in different equilibrium types. Consider a contest with four H-cost players, we have an equilibrium with symmetry, illustrated in the leftmost figure. If we replace one H-cost player with an L-cost player, the equilibrium changes to an equilibrium with embedment, illustrated in the middle-left figure. If we replace another H-cost player with an L-cost player, the equilibrium becomes the one with separation, illustrated in the middle-right figure. Finally, replacing one more H-cost player with an L-cost one, we obtain an equilibrium of nonperformance, which is illustrated in the rightmost figure.

4 Discouraging vs. Encouraging Entry

Consider an n-player contest with n_L L-cost players. Suppose an L-cost player enters and replaces an H-cost player, who is less productive. Notice that the size of contest is fixed to n players, so there is no scale effect.[11] As a result, $n-1$ players before the entry remain afterwards. In the next two sections, we study how the entry affects these $n-1$ players' equilibrium performance. In particular, Section 4.1 studies how the entry affects individual player's performance, while Section 4.2 studies how the entry affects the total performance of the other players.

4.1 Effects on Individual Performance

As we can see in Examples 1-4, as an L-cost player enters, an H-cost player's strategy G_H shifts up, so his performance after the entry first order stochastically dominates that before

[11] In many contests, prize structure changes are rare and infrequent. For instance, since the 1990s professional golf tournaments have been designating a specific percentage to each prize, from first to 70th: 18% of the total purse goes to the winner, 10.8% to the second place, ..., and 0.2% to the 70th. In addition, the "20-70-10" system was used for twenty years in General Electric from 1981 to 2001.

the entry. However, after an L-cost player enters, an L-cost player mixes over a wider interval. For example, the entry of the second L-cost player changes the support of G_L from $[1.1, 3]$ to $[1/3, 5]$, and the entry of the third L-cost player changes the support from $[1/3, 5]$ to $[0, 8]$. As a result, the before and after-entry performance of an L-cost player cannot be ranked according to first order stochastic dominance. However, they can be ranked in their mean: the before-entry performance has a lower mean than the after-entry performance. Thus, in the examples, when facing more L-cost opponents, an H-cost player reduces his expected performances, while an L-cost player increases his expected performances.

Below we study this question in general: when facing more L-cost opponents, does an L-cost player increase or decrease his expected performance? How about an H-cost player? Let G_t^{before} be the t-cost player's equilibrium strategy before the entry of an L-cost player and G_t^{after} be his equilibrium strategy after the entry. Moreover, let $[\underline{s}_t^{\text{before}}, \bar{s}_t^{\text{before}}]$ be the support of G_t^{before} and $[\underline{s}_t^{\text{after}}, \bar{s}_t^{\text{after}}]$ be the support of G_t^{after}. The following result discusses how the entry affects individual player's equilibrium strategy:

Proposition 5 *Facing one more L-cost opponent, an H-cost player's performance is less spread out: $\underline{s}_H^{before} = \underline{s}_H^{after} \leq \bar{s}_H^{after} \leq \bar{s}_H^{before}$. Moreover, G_H^{after} is first order stochastically dominated by G_H^{before}, and therefore $\mathbb{E}[s_H^{before}] > \mathbb{E}[s_H^{after}]$.*

Facing one more L-cost opponent, an L-cost player's performance is more spread out: $\underline{s}_L^{after} \leq \underline{s}_L^{before} < \bar{s}_L^{before} \leq \bar{s}_L^{after}$. Moreover, if α is sufficiently small, $\mathbb{E}[s_L^{before}] < \mathbb{E}[s_L^{after}]$. If α is sufficiently large, there is a weakly decreasing function $\gamma_\alpha(\frac{n_L}{n})$ such that $\mathbb{E}[s_L^{before}] < \mathbb{E}[s_L^{after}]$ if and only if $\frac{c_L}{c_H} < \gamma_\alpha(\frac{n_L}{n})$.

Recall that an L-cost player is relatively stronger than an H-cost player. Intuitively, facing more stronger L-cost opponents, an H-cost player expects to receive lower winnings, which leads to lower performance to compete. This effect on winnings remains for an L-cost player: facing more L-cost opponents, an L-cost player expects to win less. In the introduction, we refer to it as the disincentive-from-top effect. However, there is incentive-from-bottom effect: facing more L-cost opponents, an L-cost player expects to receive lower payoffs as well. Note that this effect is absent for H-cost players as their equilibrium payoffs are always 0. Since an L-cost player's expected performance is $\mathbb{E}[s_L] = (w_L - u_L)/c_L$, the two effects push his expected performance towards opposite directions.

It turns out that as the cost types becomes more asymmetric (smaller c_L/c_H) or as the prize sequence becomes more convex (larger α), the disincentive-from-top effect becomes stronger relative to the incentive-from-bottom effect. As a result, for large α and small c_L/c_H, the disincentive-from-top effect dominates the incentive-from-bottom effect, which leads to $\mathbb{E}[s_L^{\text{before}}] > \mathbb{E}[s_L^{\text{after}}]$. Next, we explain in an example the roles of cost asymmetry and prize convexity.

As the two cost types become more asymmetric, c_L/c_H decreases. In Example 3, the winnings of an L-cost player $w_L = \frac{v_{(4)}+v_{(3)}}{2}$ is unaffected by c_L/c_H. In contrast, $u_L = v_{(3)} - v_{(2)}c_L/c_H$, which means when facing weaker H-cost players, an L-cost player's payoff is lower. Then, more L-cost opponents has less impact on the already low payoff of u_L.

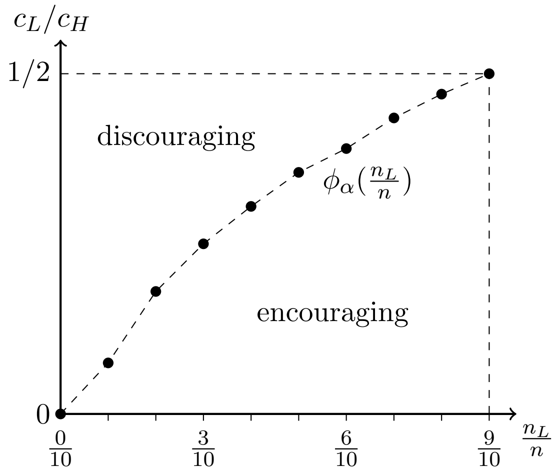

Figure 7: $\alpha = 0$

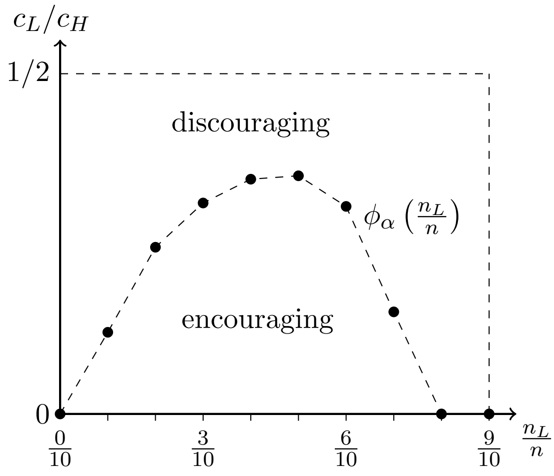

Figure 8: $\alpha = 2$

Recall that α increasing means the prize sequence becomes more convex. In Example 3, suppose α is sufficiently large, so a higher prize is disproportionally larger than the lower ones. To illustrate the main idea, we ignore lower prizes in the presence of a higher one. The expected winnings $w_L^{\text{before}} = \frac{v_{(4)}+v_{(3)}}{2} \approx \frac{v_{(4)}}{2}$ reduces to $w_L^{\text{after}} = \frac{v_{(4)}+v_{(3)}+v_{(2)}}{3} \approx \frac{v_{(4)}}{3}$. In contrast, the equilibrium payoff $u_L^{\text{before}} \approx v_{(3)}$ reduces to $u_L^{\text{after}} \approx v_{(2)}$. As α increases, the prize sequence becomes more convex, and the winnings effect $w_L^{\text{after}} - w_L^{\text{before}} \approx \frac{v_{(4)}}{6}$ increases faster than the payoff effect $u_L^{\text{after}} - u_L^{\text{before}} \approx v_{(3)} - v_{(2)}$ does.

4.2 Effects on Total Performance of Other Players

In Section 4.1, we study how the entry of an L-cost player affects individual player's performance level, and find that the entry may increase one player's performance while decreasing another's. Recall that $n-1$ players remain after the entry. In this section, we study the entry's effect on the $n-1$ players' total performance. More precisely, does the entry of an L-cost player decrease the total expected performance of the others? Notice that the change caused by the entry is equivalent to reducing one player's marginal cost from c_H to c_L while fixing the marginal costs of the others. Therefore, we can rephrase the question: Does the other players' total performance decreases when a player's marginal cost decreases?

We say a contest is *discouraging* if a player's lower marginal cost leads to lower total expected equilibrium performance from others. A contest is *encouraging* if a player's lower marginal cost leads to higher total expected equilibrium performance from others.

So far we have been discussing the main question in the context of entry/hiring. The question is also relevant in decisions on staff training. For example, if an employee applies for a training course, should the company approve it? If it approves, the training reduces the performance cost of the employee. Then, does this change discourage his colleagues' overall performance, or encourage their performance? Proposition 1 in Section 4 answers these questions by characterizing the condition on $(\alpha, n_L/n, c_L/c_H)$ such that a contest is discouraging/encouraging.

Proposition 6 *Fix the prize sequence and participant composition, the entry of an L-cost player is encouraging if and only if the difference between cost types is sufficiently large. That is, given any α and n_L/n, there exists a unique $\phi_\alpha(n_L) \in \mathbb{R}_+$ such that $\Pi^{before}_{other}(n_L) < \Pi^{after}_{other}(n_L)$ if and only if $\frac{c_L}{c_H} < \phi_\alpha(\frac{n_L}{n})$.*

If we fix the composition of a contest, Proposition 6 implies that small c_L/c_H leads to the encouraging effect. Let us explain why. If an L-cost player enters the contest, he encourages the existing L-cost players and discourages the H-cost players. If c_L/c_H decreases, the L-cost players becomes more productive than the H-cost ones. Then, a larger share of the performance comes from the L-cost players, so the encouraging effect among the L-cost players becomes more pronounced than the discouraging effect among the H-cost players. Hence, if c_L/c_H is sufficiently small, the encouraging effect dominates the discouraging effect. It turns out $\phi_\alpha(0) = 0$ (see Lemma 2 in Appendix C), which means if an L-cost player enters a contest containing only H-cost players, the entry must have the discouraging effect.

Proposition 7 *If the prize sequence is not very convex, a contest with more weak participants is more likely to be discouraging. That is, $\phi_\alpha(\frac{n_L}{n})$ is increasing in $\frac{n_L}{n}$ if α is small.*

If the prize sequence is sufficiently convex, a contest with more homogeneous participants is more likely to be discouraging. That is, $\phi_\alpha(\frac{n_L}{n})$ is hump-shaped in $\frac{n_L}{n}$ if α is large enough.

Figure 7 illustrates the first half of Proposition 7 if $n = 10$ and $\alpha = 0$. As in the figure, the entry of a strong player increases the participants' performance if they are relatively strong, and decreases their performance if they are relatively weak. To see why, recall that the entry of an L-cost player encourages the other L-cost players while discouraging the H-cost players. Then, as n_L/n increases, the encouraging effect is more pronounced because there are more L-cost players.

In contrast, Figure 8 illustrates the second half of Proposition 7 if $n = 10$ and $\alpha = 2$. As in the figure, the entry of a strong player increases the participants' performance if they are more heterogeneous, and decreases their performance if they are more homogeneous. Let us explain why the convexity of the prize sequence leads to the difference between the results in Proposition 7. Consider an extreme case with $\alpha \to \infty$, so the other prizes are negligible compared to the first one. If there are n_L L-cost players before the entry, they are competing for the first prize and each wins the prize with probability $1/n_L$. After the entry, there is one more L-cost player, so each of them win with a lower probability $1/(n_L + 1)$, which discourages their performance. Moreover, this effect is more pronounced if the prize sequence is more convex.

Proposition 7 demonstrates two possible shapes of ϕ_α. There may be other shapes. Example 5 in the Appendix C illustrates an N-shaped ϕ_α.

5 Barriers to All-Star Contests

In many situations, contest organizers want to improve participants' ability. For example, a sport tournament wants to attract better athletes, an academic department wants to recruit more productive researchers, and a company wants to improve its employees' productivity. In this section, we study barriers a contest organizer faces when she tries to improve participants' ability. Specifically, can a company improve its employee ability without suffering from the discouraging effect? To answer this question, we start with a company with n employees. Among them, n_L are high-ability and have the low marginal cost c_L, and $n - n_L$ of them are of low-ability and have the high marginal cost c_H. The employees compete in an internal contest described in Section 2. The company improves its employees' ability, one employee at a time. As a result, the number of high-ability employees, n_L increases while the number of low-ability ones, $n - n_L$ decreases. Eventually, we reach an *all-star* contest, in which all the employees are of high-ability. In the above process, can the company achieve an all-star contest without suffering from the discouraging effect?

The answer to the above question depends on the proportion of high-ability employees n_L/n, and the convexity of the prize sequence α. It is easy to see in the following two examples. First, suppose $\alpha = 0$, $n = 10$ and $n_L = 3$. In Figure 9, for a given c_L/c_H, the effects of improving employee ability are represented by the horizontal arrows. Moreover, ability improvements suffering from the discouraging effect are represented by red arrows, and those without the discouraging effect are represented by blue arrows. In the figure, if c_L/c_H is sufficiently low, improving an employee's ability always encourages the other players. In contrast, if c_L/c_H is not low enough, improving an employee's ability first discourages the others then encourages them. Thus, if the prize sequence is not too convex, a sufficiently high proportion of high-ability employees and sufficiently large difference between ability types ensure the all-star contest can be reached without the discouraging effect.

Second, suppose $\alpha = 2$, $n = 10$ and $n_L = 3$. As we can see in Figure 10, if c_L/c_H is sufficiently high, improving an employee's ability always discourages the others. In contrast, if if c_L/c_H is sufficiently low, improving an employee's ability first encourages the others then discourages them. In both cases, the all-star contests cannot be reached without suffering from the discouraging effect. In the latter case, the company can still improve its employees' ability but cannot make all of them high-ability.

Corollary 2 *Suppose the prize sequence is not very convex. Improving an employee's ability first discourages the others when the high-ability proportion is low, but eventually encourages the others as the high-ability proportion becomes sufficiently high.*

Suppose the prize sequence is sufficiently convex. Improving an employee's ability first may encourage the others when the high-ability proportion is low, but eventually discourages the others as the high-ability proportion becomes sufficiently high.

Since the corollary is straightforward implication of Proposition 7, we omit its proof. According to the corollary, a very convex prize sequence imposes barriers for the company to improve all employees to high-ability. In particular, if the ability improvement is sufficiently large (sufficiently low c_L/c_H) and there are enough high-ability employees, the company may be able to improve employees' ability without discouraging the others. However, as the high-ability

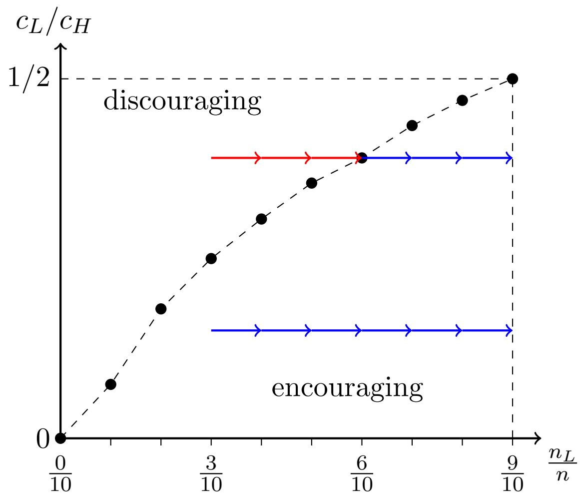

Figure 9: $\alpha = 0$

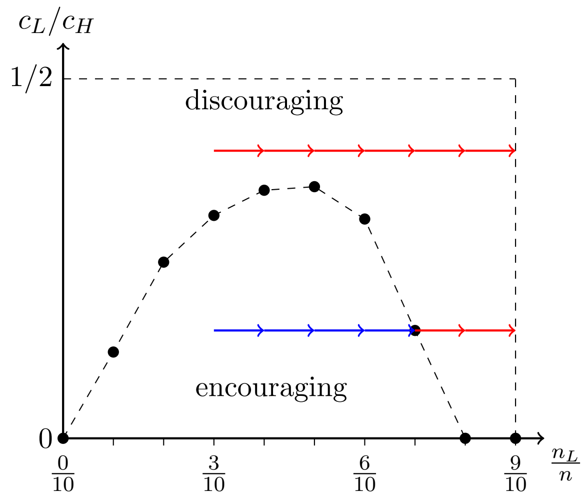

Figure 10: $\alpha = 2$

proportion increases, the company eventually hits the barrier and suffers from the discouraging effect.

In contrast, contests with a less convex prize sequence may not have such barriers. If the ability improvement is sufficient large and the high-ability proportion is high enough, the company can keep improving employee ability until all of them are high-ability. Thus, as the proportion of its high-ability employees grows, the company needs to make its prize sequence less convex in order to avoid the discouraging effect.

6 Effects on Total Performance

In Section 4 we study the entry of an L-cost player on individual performance and the total performance of the other players. If the entry discourages the other players, can the entrant's higher performance make up the reduced performance from the others? To answer this question, we study below the effect of an L-cost player's entry on the total performance of all players in the contest.

According to Proposition 6, if $\frac{c_L}{c_H} < \phi_\alpha(\frac{n_L}{n})$, the entry of an L-cost player increases the other players' total performance. Moreover, this L-cost player replaces an H-cost player. Therefore, if the entry has the encouraging effect on the other players, the entry results in higher total performance. This result is formalized in the proposition below.

Proposition 8 *If $c_L/c_H \le \phi_\alpha(n_L/n)$, the entry of an L-cost player increases the total expected performance.*

If $c_L/c_H > \phi_\alpha(n_L/n)$, the entry of an L-cost player reduces other players' total performance, which puts a downward pressure on the total performance. In contrast, this L-cost player replaces an H-cost player, so the lower marginal performance cost puts an upward pressure on the total performance. Thus, it is not obvious whether the total expected performance is higher after an L-cost player enters.

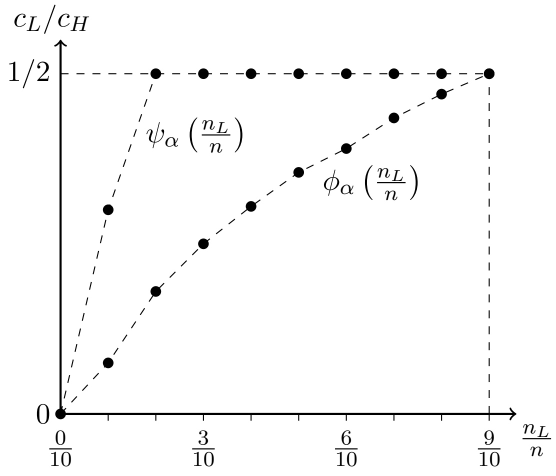

Figure 11: Entry Effects if $\alpha = 0$

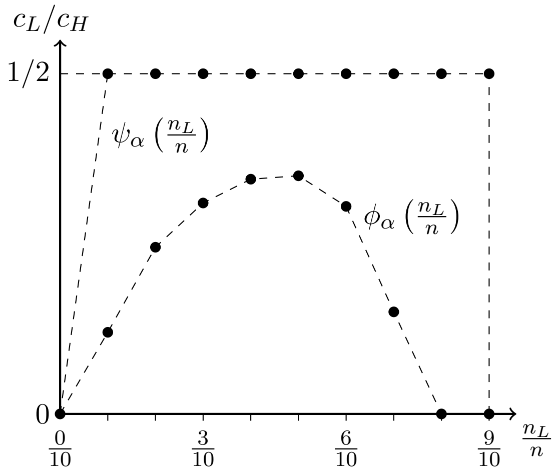

Figure 12: Entry Effects if $\alpha = 2$

Let $\Pi_{\text{all}}(n_L)$ be the total performance when there are n_L L-cost participants. After the entry of another L-cost player, the total performance becomes $\Pi_{\text{all}}(n_L + 1)$. The result below characterizes the condition for the entry to improve the total performance, as well as the condition for it to reduce the total performance.

Proposition 9 *The entry of an L-cost player results in higher total expected performance if and only if the contest has sufficiently high fraction of L-cost players. For any α, there exists a mapping $\psi_\alpha : \{\frac{0}{n}, \frac{1}{n}, ..., \frac{n-1}{n}\} \to [0, 1/2]$ such that $\Pi_{all}(n_L + 1)/\Pi_{all}(n_L) < 1$ if and only if $\frac{c_L}{c_H} > \psi_\alpha(\frac{n_L}{n})$. Moreover, ψ_α is nondecreasing and $\psi_\alpha(\frac{n_L}{n}) \geq \phi_\alpha(\frac{n_L}{n})$.*

Figures 11 and 12 illustrate the mapping ψ_α. In both figures, $\psi_\alpha(\frac{n_L}{n})$ lies above $\phi_\alpha(\frac{n_L}{n})$. As a result, if the two performance costs are sufficiently different so that $\frac{c_L}{c_H} < \phi_\alpha(\frac{n_L}{n})$, the entry encourages the other players and increases the total performance. If the performance costs become more similar so that $\phi_\alpha(\frac{n_L}{n}) < \frac{c_L}{c_H} < \psi_\alpha(\frac{n_L}{n})$, the entry discourages the other players but still increases the total performance. This is because the performance from the entrant makes up the reduced performance from the others. If the performance costs become sufficiently similar so that $\frac{c_L}{c_H} > \psi_\alpha(\frac{n_L}{n})$, the entrant's high performance is not enough to make up the others' low performance, so the entry discourages the other players and decreases the total performance.

In addition, notice that in the figures the function $\phi_\alpha(\frac{n_L}{n})$, which divides the encouraging effect and discouraging effect, may be increasing or hump-shaped. In contrast, the function $\psi_\alpha(\frac{n_L}{n})$, which divides the increased total performance and decreased total performance, is always weakly increasing. We explain the intuition below.

An important reason for $\phi_\alpha(\frac{n_L}{n})$ to be hump-shaped is the disincentive-from-top effect. Recall that as another L-cost player enters, an existing L-cost player needs to share the top prizes with one more player, which discourages his performance. However, for consideration of total performance, this effect is endogenized because the loss of one L-cost player is the gain of another L-cost player. Thus, the disincentive-from-top effect does not apply to the total performance. As a result, due to the incentive-from-bottom effect, the entry increases each L-cost player's performance but decreases each H-cost player's performance. Hence, the more L-cost players in the contest, the more likely that the entry increases the total performance, that is, $\phi_\alpha(\frac{n_L}{n})$ (weakly) increases in $\frac{n_L}{n}$.

7 Literature

Many firms use internal rank-competition to incentivize employees. In the 1980s, Jack Welch, then Chief Executive of US company General Electric, introduced a "20-70-10" system to motivate the employees, where those with top 20% evaluations should be generously awarded and those at bottom 10% should be fired. Other firms using interval rank-competition include Amazon, General Motors, Hewlett-Packard, IBM, Johnson&Johnson, Motorola and Yahoo.[12]

As discussed in the introduction, empirical studies demonstrate the encouraging and discouraging effects in different contests. In theoretical work on contests, and closely related auctions, it is also the case that different effects are demonstrated in different setups. The encouraging/discouraging effects illustrated in studies on favoritism/handicap in contests are similar to ours in the case of not very convex prize sequences. For example, in two-player Tullock contests (e.g. Baik 1994, Nti 1999) or two-player complete-information all-pay auctions (e.g. Konrad 2002, Fu 2006), leveling the playing field, or making the weaker player stronger or the stronger player weaker, can improve effort. Similar results are illustrated in two-player incomplete-information all-pay auctions as well (e.g. Amann and Leininger 1996, Lizzeri and Persico 2000). According to these results, the entry of a high-ability player discourages the low-ability player while encourages the high-ability one. Our findings generalize these effects to n-player contests. Obviously, in two-player auctions, there cannot be a convex prize sequence, so our findings for sufficiently convex prize sequences do not have counterparts in two-player contests.

Discouragement effect is illustrated in contests with a single prize. For example, Baye, Kovenock and de Vries (1993) study a complete-information all-pay auction, which is isomorphic to a contest with a single prize, with more than two players. A single prize can be viewed as an extremely convex prize sequence, so their discouraging effect is similar to our findings with sufficiently convex prize sequences. Our setup allows prize sequences with different convexity. In the theoretical section of the paper, Brown (2011) shows a superstar's entry discourages the others' performance in two Tullock contests. First, if a high-ability player enters a contest of one prize and n low-ability players, the low-ability players reduce their performance. Second, if a high ability player enters a contest with two prizes and three players with different abilities, the other players also reduce their performance. The analysis in the three-player contest is based on numerical solutions because the equilibrium has no closed-form characterization. Table 1

[12] See Peters and Waterman (1988) for rank-competitions in IBM, Johnson&Johnson, General Motor, Hewlett-Packard; Caroll and Tomas (1995) for Motorola; Swisher (2013) for Yahoo and Kantor and Streitfeld (2015) for Amazon.

	Players	Prizes	Entry Effects
Bay et al. (1993)	one strong and n weak	one prize	discouraging
Baik (1994), Nti (1999), Fu (2006)	one strong and one weak	one prize	encouraging
Brown (2011)	one strong and n weak	one prize	discouraging
	3 different players	two prizes	
This paper	n_L strong and $n - n_L$ weak	n prizes	encouraging discouraging

Table 1: Discouraging/Encouraging Effects in Literature

summarizes the relationship of this paper and the above literature.

This paper provides a tractable model that accommodates prize sequences with different convexity, and arbitrary composition of the two ability types, and players of heterogenous abilities. Such contests with asymmetric players and heterogeneous prizes are notoriously known for complex equilibrium strategies, and only a few models turn out to be tractable. For example, with asymmetric participants, equilibrium characterization is provided in contests with two heterogeneous prizes (e.g. Moldovanu and Sela 2001, Szymanski Valletti 2005, Xiao 2018), with a linear prize sequence with $\alpha = 0$ (e.g. Bulow and Levin 2006, González-Díaz and Siegel 2013), with prizes of identical values (e.g. Siegel 2010). The prize sequences in this paper generalize the linear prize sequence and allow $\alpha \geq 0$, so they can have different levels of convexity. Xiao (2016) studies geometric prize sequence, in which consecutive prizes have the same ratio, and quadratic prize sequences, which include the prize sequences in this paper as special cases. Moreover, Xiao (2016) studies participants whose abilities are all different, and characterizes equilibrium strategies as solutions to ordinary differential equations. In contrast, participants in this paper have two ability types, and we explicitly solve the equilibrium strategies. With a large number of participants, Olszewski and Siegel (2016) study contests with a general distribution of prize value and participant ability and solve equilibrium strategies explicitly. However, the entry effects in this paper vanish in large contests because the entry does not change the ability distribution.

This paper studies how a high-ability opponent's entry effects the other participants' performance. Many other features affecting participants' performance choices are also studied. For example, in dynamic elimination tournaments, Brown and Minor (2014) study how the winning probability varies with the ability of future competitor and effort spillovers across stages. In a repeated contest, Kubitz (2017) studies how early actions affect the belief of competitors' ability and therefore affect effort choices. In our paper, we do not have private information, so those reasons do not arise. Instead of the entry effect studied in this paper, Morgan, Sisak and Várdy (2018) study participants' strategic choice between contests different in show-up fees, number or value of prizes. Fang, Noe and Strack (2019) consider how the change of competitiveness affects participants' performance, while in this paper we study how the change of participants' ability composition affect their performance.

8 Conclusion and Extensions

In this paper, we study a class of contests that may differ in three dimensions: the convexity of prize sequences, the composition of ability types, and difference between ability types. We provide a closed-form characterization of the unique type-symmetric Nash equilibrium for such contests. Using the characterization, we consider which contests, in the three-dimensional space, are encouraging/discouraging so that the entry of a high-ability player increases/decreases the performance of other participants.

We consider the pure competitive effect, which means the participants only view others as opponents and a participant's presence does not affect others' ability. It is an interesting extension to consider externalities among the players. The externalities are mutual in the sense that the low-ability players may have a negative externality on the high-ability one and the high-ability players may have a positive externality on the low-ability players. Then, it is unclear how externalities affect the encouraging and discouraging effects identified in this paper. Our model could serve as a benchmark to introduce externalities, and we hope to explore this idea in future research.

The model in this paper has *complete* information. While it would be desirable to study a similar environment under *incomplete* information, the problems associated with multiple prizes and asymmetric players under incomplete information are well known from auction theory. For instance, even with symmetric players very little is known about discriminatory (pay-as-you-bid) auctions for the sale of multiple units. Similar difficulties arise when considering all-pay auctions with multiple prizes.[13] The complete information setting allows us to study environments that, as yet, cannot be studied under an incomplete information setting.

[13]Studies of similar cases have shown that there is a unique equilibrium in asymmetric all-pay auctions with two players (Amann and Leininger 1996, Lizzeri and Persico 2000). The complexity in the case of more than two players is demonstrated by Parreiras and Rubinchik (2010).

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Appendices

A Techniques from Combinatorics

In this section, we present a selection of results in Combinatorics (see, for example, Riordan 2012), and use them to derive properties of our model that we repeatedly use in the proofs.

The **Pascal's identity**: $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$. Recall that $v_{(k)} = C_1^{k-1} + \alpha C_2^{k-1}$, so

$$\Delta_k^1 = v_{(k+1)} - v_{(k)} = C_1^k - C_1^{k-1} + \alpha(C_2^k - C_2^{k-1}) = C_0^{k-1} + \alpha C_1^{k-1} \tag{15}$$

where the last equality is from the Pascal's identity.

The **hockey-stick identity**: $\sum_{m=k}^{n} C_k^m = C_{k+1}^{n+1}$. The sum of all the prizes is

$$V = \sum_{k=1}^{n} v_{(k)} = \sum_{k=1}^{n} (C_1^{k-1} + \alpha C_2^{k-1}) = C_2^n + \alpha C_3^n \tag{16}$$

where the last equality is from the hockey-stick identity.

The **absorption identity**: $C_k^n = \frac{n}{k} C_{k-1}^{n-1}$, so $\frac{C_k^n}{n} = \frac{C_{k-1}^{n-1}}{k}$. Then, the average value of prizes is

$$V/n = \frac{C_2^n + \alpha C_3^n}{n} = \frac{C_1^{n-1}}{2} + \frac{\alpha C_2^{n-1}}{3} \tag{17}$$

where the last equality is from the absorption identity. Similarly, the average of the lowest m prizes is

$$\frac{\sum_{k=1}^{m} v_{(k)}}{m} = \frac{C_1^{m-1}}{2} + \frac{\alpha C_2^{m-1}}{3} \tag{18}$$

and the average of the top m prizes is

$$\begin{aligned} \frac{\sum_{k=n-m+1}^{n} v_{(k)}}{m} &= \frac{V - \sum_{k=1}^{n-m} v_{(k)}}{m} \\ &= \frac{C_2^n - C_2^{n-m} + \alpha(C_3^n - C_3^{n-m})}{m} \\ &= \frac{2n-m-1}{2} + \alpha\frac{n(n-1) + (2n-m-1)(n-m-2)}{6} \end{aligned} \tag{19}$$

where the second equality is from (16).

B Omitted Proofs in Section 3

Proof of Proposition 2. We first verify that $G_H(s)$ given in (7) and (8) is strictly increasing and has values in $[0,1]$. Using the closed-form expressions in (7) and (8), it is straightforward to verify that G_H is continuous, strictly increasing, and $G_H(s) \in [0,1]$ for $s \in [0, \frac{v_{(n)}}{c_H}]$.

Next, we verify that G_L is continuous, strictly increasing, and has values in $[0,1]$. Since

$G_L(\frac{v_{(n)}}{c_H}) = 1$, it is sufficient to show that $G'_L(s) > G'_H(s)$ whenever $G_L(s) \geq 0$. Recall that the strategies in (8) and (9) are solution to

$$\begin{aligned} \alpha C_2^{n-1} G_H^2 + C_1^{n-1} G_H - c_L s &= u_L \\ \alpha(C_2^{n-2} G_H^2 + C_1^{n-2} G_H G_L) + (C_1^{n-2} G_H + G_L) - c_H s &= 0 \end{aligned} \tag{20}$$

Taking derivative with respect to s, the above equation system becomes

$$\alpha C_2^{n-1} 2 G_H g_H + C_1^{n-1} g_H = c_L \tag{21}$$

$$\alpha[C_2^{n-2} 2 G_H g_H + C_1^{n-2}(g_H G_L + G_H g_L)] + (C_1^{n-2} g_H + g_L) = c_H \tag{22}$$

where $g_t(s)$ is the derivative of $G_t(s)$. We claim that $g_H(s) < g_L(s)$ if $G_H(s) \geq G_L(s)$. To see this, suppose otherwise that $g_H(s) \geq g_L(s)$ and $G_H(s) \geq G_L(s)$ for some s. Then, the two inequalities implies that the left hand side of (22) is lower or equal to

$$\begin{aligned} & \alpha[2C_2^{n-2} + 2C_1^{n-2}] G_H g_H + (C_1^{n-2} + 1) g_H \\ = \; & \alpha C_2^{n-1} 2 G_H g_H + C_1^{n-1} g_H = c_L < c_H \end{aligned}$$

where the second equality is from (21). Therefore, (22) cannot hold, which is a contradiction. Thus, we have $g_H(s) < g_L(s)$ if $G_H(s) \geq G_L(s)$. Since $G_H(v_{(n)}) = G_L(v_{(n)})$, we have $g_H(v_{(n)}) < g_L(v_{(n)})$, which means $G_H(s) > G_L(s)$ for s slightly below $v_{(n)}$. Moreover, $g_H(s) < g_L(s)$ implies that the difference between $G_H(s)$ and $G_L(s)$ is increasing. Hence, we always have $G_H(s) \geq G_L(s)$, and therefore $g_H(s) < g_L(s)$.

Let $\bar{s}_H$ be the highest performance in the support of G_H. Since the L-cost player can always choose $\bar{s}_H$ at a lower cost than an H-cost player can, his equilibrium payoff is $u_L > u_H$.

Next, we argue that G_H has no atom, which means G_H does not assign positive probability to any $s \geq 0$. To see this, suppose G_H has an atom at $s \geq 0$. Then, given others' equilibrium strategies, an H-cost player's payoff increases discontinuously as he increases his performance from s to slightly above. This means s is not a best response, which contradicts the assumption that G_H assigns positive probability to it.

We show below that $\underline{s}_L > 0$, where $\underline{s}_L$ is the lowest performance in the support of G_L. Suppose otherwise that $\underline{s}_L = 0$. Then, given others' equilibrium strategies G_H, by choosing $\underline{s}_L$ an L-cost player receives the lowest prize $v_{(1)} = 0$. This is because G_H has no atom at 0. Therefore, the L-cost player's payoff is $u_L = 0$, which contradicts with $u_L > u_H$.

Then, we must have $\underline{s}_H = 0$. Suppose otherwise that $\underline{s}_H > 0$. If an H-cost player deviates from s_H to 0, he receives the same expected prize at a lower cost, which can never arise in an equilibrium.

Given others' equilibrium strategies, an H-cost player receives $v_{(1)} = 0$ prize with certainty by choosing $s = 0$. This is because $\underline{s}_L > 0$ and G_H has no atom. Thus, $u_H = 0$.

Next, we show $u_L = v_{(n)}(1 - c_L/c_H)$. Notice that $\bar{s}_H \leq v_{(n)}/c_H$ because any performance above $v_{(n)}/c_H$ can only give an H-cost player a negative payoff. By choosing $\bar{s}_H$, an L-cost player can guarantee himself a payoff of $v_{(n)}(1 - c_L/c_H)$. Thus, $u_L \geq v_{(n)}(1 - c_L/c_H)$. Suppose the inequality is strict: $u_L > v_{(n)}(1 - c_L/c_H)$. Then, $\bar{s}_L < v_{(n)}/c_H$. In addition, we must have $\bar{s}_H = v_{(n)}/c_H$ because $\bar{s}_H < v_{(n)}/c_H$ would imply $u_H > 0$. As a result, only H-cost players mix over $[\bar{s}_L, \bar{s}_H]$ and their strategy $\hat{G}_H$ satisfies

$$\alpha C_2^{n-2}\hat{G}_H^2 + (1+\alpha)C_1^{n-2}\hat{G}_H + 1 - c_H s = 0 \tag{23}$$

To see why, recall that G_H in (8) is a solution to (20). Replacing G_L with 1 in (20) gives us (23). For s slightly below $v_{(n)}/c_H$, we have $G_L(s) < 1$. Since the left hand side in (20) is increasing in both G_L and G_H, as we increase G_L to 1 in (20), its solution decreases from $G_H(s)$ to $\hat{G}_H(s)$. Thus, at any s slightly below $v_{(n)}/c_H$, the L-cost player's payoff is $\alpha C_2^{n-1}\hat{G}_H^2 + C_1^{n-1}\hat{G}_H - c_L s$, which is strictly lower than $\alpha C_2^{n-1}G_H^2 + C_1^{n-1}G_H - c_L s = v_{(n)}(1 - c_L/c_H)$. This is a contradiction to $u_L \geq v_{(n)}(1 - c_L/c_H)$ established above. Thus, $u_L = v_{(n)}(1 - c_L/c_H)$.

We also show above that $\bar{s}_L < v_{(n)}/c_H$ leads to a contradiction, so $\bar{s}_L = v_{(n)}/c_H$. Then, $\bar{s}_H = v_{(n)}/c_H$, otherwise the L-cost player could benefit from reducing $\bar{s}_L$.

It remains to show that both G_H and G_L have interval supports. To see this, suppose G_H has a gap in its support: there is an interval (s'_H, s''_H) such that $G_H(s'_H) = G_H(s''_H) \in (0,1)$. Then, the L-cost player is the only one mixing over that gap, which violates the property of no aggregate gaps. Suppose G_L has a gap (s'_L, s''_L) in its support, then $G_L(s) = G_L(s'_L)$ for all $s \in [s'_L, s''_L]$. Given others' equilibrium strategies, an H-cost player's payoff by choosing any $s \in (s'_L, s''_L)$ is

$$\alpha(C_2^{n-2}G_H^2 + C_1^{n-2}G_H G_L(s'_L)) + (C_1^{n-2}G_H + G_L(s'_L)) - c_H s = 0 \tag{24}$$

Let $\hat{G}_H(s)$ is the solution to this equation. Since $G_L(s)$ in (9) is lower than $G_L(s'_L)$ for $s \in [s'_L, s''_L)$, the solution $G_H(s)$ of (8) is higher than that of (24), i.e. $G_H(s) > \hat{G}_H(s)$ for $s \in [s'_L, s''_L)$. Hence, given others' strategies, an L-cost player's payoff by choosing s'_L is strictly lower than u_L, which is a contradiction. ■

Proof of Proposition 3. Using (11) and (13), it is straightforward to verify that $G_H(s)$ and $G_L(s)$ are strictly increasing and continuous in s, and are between $[0,1]$. If others use strategies given in (11) and (13), an H-cost player's payoff by choosing any $s > \frac{v_{(n_H)}}{c_H}$ is

$$U_H(s|G_H, G_L) = \alpha C_2^{n_L} G_L^2(s) + C_1^{n_L}\Delta^1_{n_H} G_L(s) + v_{(n_H)} - c_H s \tag{25}$$

We show in the next two paragraphs that this payoff is always negative.

We first verify that $U_H(s|G_H, G_L)$ is convex in s. If $n_L = 2$, $G_L(s)$ is linear in s, so substituting (13) into (25) we obtain $U_H(s|G_H, G_L)$ as a quadratic function. It is straightforward to verify the quadratic function is convex. If $n_L > 2$, recall that $G_L(s)$ is a solution to (12), from which we obtain

$$\alpha G_L^2(s) = \frac{-c_L v_{(n_H)}/c_H + c_L s - \Delta^1_{n_H+1} C_1^{n_L-1} G_L(s)}{C_2^{n_L-1}}$$

Substituting this into (25), we obtain

$$U_H(s|G_H, G_L) = C_2^{n_L} \frac{-c_L v_{(n_H)}/c_H + c_L s - \Delta^1_{n_H+1} C_1^{n_L-1} G_L(s)}{C_2^{n_L-1}} + C_1^{n_L} \Delta^1_{n_H} G_L(s) + v_{(n_H)} - c_H s$$

whose derivative is

$$\begin{aligned} U_H'(s|G_H, G_L) &= \left(C_1^{n_L} \Delta^1_{n_H} - \frac{C_2^{n_L} C_1^{n_L-1}}{C_2^{n_L-1}} \Delta^1_{n_H+1} \right) g_L(s) + \frac{C_2^{n_L}}{C_2^{n_L-1}} c_L - c_H \\ &= n_L \left(\Delta^1_{n_H} - \frac{n_L - 1}{n_L - 2} \Delta^1_{n_H+1} \right) g_L(s) + \frac{C_2^{n_L}}{C_2^{n_L-1}} c_L - c_H \end{aligned} \tag{26}$$

Notice that $\Delta^1_{n_H} < \Delta^1_{n_H+1}$, so $\Delta^1_{n_H} - \frac{n_L-1}{n_L-2}\Delta^1_{n_H+1} < 0$. In addition, using (13), it is straightforward to verify that $g_L(s)$ decreases in s. Thus, (26) implies that $U_H'(s|G_H, G_L)$ increases in s. Therefore, $U_H(s|G_H, G_L)$ is also convex in s if $n_L > 2$.

Then, we show that $U_H(s|G_H, G_L) < 0$ at $s = \frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}$. If an H-cost player chooses performance $\frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}$, his payoff is no more than

$$\begin{aligned} & v_{(n)} - v_{(n_H)} - (v_{(n)} - v_{(n_H+1)}) \frac{c_H}{c_L} \\ < \; & v_{(n)} - v_{(n_H)} - 2(v_{(n)} - v_{(n_H+1)}) \\ \leq \; & -(v_{(n_H+2)} - v_{(n_H+1)}) + (v_{(n_H+1)} - v_{n_H}) \\ \leq \; & -\alpha \end{aligned}$$

where the first inequality is from $c_H > 2c_L$. Recall that $\alpha \geq 0$, so $U_H(s|G_H, G_L) < 0$ for $s = \frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}$. It is straightforward to verify that $U_H(\frac{v_{(n_H)}}{c_H}|G_H, G_L) = 0$. Hence, the convexity shown above implies $U_H(s|G_H, G_L) < 0$ for any $s \in (\frac{v_{(n_H)}}{c_H}, \frac{v_{(n)} - v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}]$.

In the next two paragraphs, we show $u_L \geq v_{(n_H+1)} - v_{(n_H)} \frac{c_L}{c_H}$. Suppose otherwise that $u_L < v_{(n_H+1)} - v_{(n_H)} \frac{c_L}{c_H}$, then we claim $\bar{s}_L > v_{(n)}/c_H$. To see this, notice that if $\bar{s}_L \geq v_{(n)}/c_H$, an L-cost player can deviate to performance $v_{(n)} c_H$ and wins $v_{(n)}$, so his equilibrium payoff satisfies

$$\begin{aligned} u_L &\geq v_{(n)} \left(1 - \frac{c_L}{c_H}\right) \\ &= \left(1 - \frac{c_L}{c_H}\right) (v_{(n)} - v_{(n_H+1)}) - \frac{c_L}{c_H} (v_{(n_H+1)} - v_{(n_H)}) + v_{(n_H+1)} - v_{(n_H)} \frac{c_L}{c_H} \\ &> \frac{1}{2} [(v_{(n)} - v_{(n_H+1)}) - (v_{(n_H+1)} - v_{(n_H)})] + v_{(n_H+1)} - v_{(n_H)} \frac{c_L}{c_H} \\ &\geq v_{(n_H+1)} - v_{(n_H)} \frac{c_L}{c_H} \end{aligned}$$

which contradicts with $u_L < v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$. Thus, $\bar{s}_L > v_{(n)}/c_H$.

Notice that the H-cost players never choose performance above $v_{(n)}/c_H$, so $\bar{s}_H \leq v_{(n)}/c_H$. In addition, we show above that $\bar{s}_L > v_{(n)}/c_H$, so $\bar{s}_L > \bar{s}_H$. Hence, only the L-cost players mix over $(\bar{s}_H, \bar{s}_L)$, so their equilibrium strategy $\hat{G}_L$ satisfies

$$\alpha C_2^{n_L-1}\hat{G}_L^2 + \Delta^1_{n_H+1}C_1^{n_L-1}\hat{G}_L + v_{(n_H+1)} - c_L s = u_L$$

Recall that we assume $u_L < v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$, so comparing the above equation with (12), we have $\hat{G}_L(s) < G_L(s)$, which is described in (13). Recall that $G_L(s)$ reaches zero at $s = \frac{v_{(n_H)}}{c_H}$, so the equilibrium strategy $\hat{G}_L$ must reach zero at a higher performance: $\underline{s}_L > \frac{v_{(n_H)}}{c_H}$. Therefore, $\bar{s}_H \geq \underline{s}_L > \frac{v_{(n_H)}}{c_H}$, where $\bar{s}_H$ is the highest performance in the support of H-cost players' equilibrium strategy. This means $G_L(\bar{s}_H)$ is well-defined. Then, given others' equilibrium strategies, an H-cost player's payoff by choosing $\bar{s}_H$ is:

$$\begin{aligned} & \alpha C_2^{n_L}\hat{G}_L^2(\bar{s}_H) + C_1^{n_L}\Delta^1_{n_H}\hat{G}_L(\bar{s}_H) + v_{(n_H)} - c_H\bar{s}_H \\ < \; & \alpha C_2^{n_L}G_L^2(\bar{s}_H) + C_1^{n_L}\Delta^1_{n_H}G_L(\bar{s}_H) + v_{(n_H)} - c_H\bar{s}_H \\ = \; & U_H(\bar{s}_H|G_H, G_L) < 0 \end{aligned}$$

where the last inequality is due to $U_H(s|G_H, G_L) < 0$ for any $s \in (\frac{v_{(n_H)}}{c_H}, \frac{v_{(n)}-v_{(n_H+1)}}{c_L} + \frac{v_{(n_H)}}{c_H}]$. This is a contradiction to the equilibrium payoff $u_H = 0$. Thus, we have $u_L \geq v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$.

We show in the next two paragraphs that $\underline{s}_L = \bar{s}_H$. If others using strategies given in (11) and (13), an L-cost player's payoff by choosing any $s \in [0, \frac{v_{(n_H)}}{c_H}]$ is

$$U_L(s|G_H, G_L) = \alpha C_2^{n_H}G_H^2(s) + C_1^{n_H}G_H(s) - c_L s$$

with $G_H(s)$ given in (11). By using the same argument for $U_H(s|G_H, G_L)$, we can show that $U_L(s|G_H, G_L)$ is convex over the interval $[0, \frac{v_{(n_H)}}{c_H}]$. In addition, at the boundaries of the interval, we have $U_L(0|G_H, G_L) < v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$ and $U_L\left(\frac{v_{(n_H)}}{c_H}|G_H, G_L\right) = v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$. Hence, $U_L(s|G_H, G_L) < v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$ for all $s \in [0, \frac{v_{(n_H)}}{c_H}]$.

Recall that $\underline{s}_H < \underline{s}_L$, so for $s \in [0, \underline{s}_L]$, the H-cost players' equilibrium strategy is given by (11). Moreover, given others' equilibrium strategies, an L-cost player's payoff by choosing performance $\underline{s}_L$ is u_L:

$$\alpha C_2^{n_H}G_H^2(\underline{s}_L) + C_1^{n_H}G_H(\underline{s}_L) - c_L\underline{s}_L = u_L$$

Recall that $u_L \geq v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$, so we have

$$\alpha C_2^{n_H}G_H^2(\underline{s}_L) + C_1^{n_H}G_H(\underline{s}_L) - c_L\underline{s}_L \geq v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$$

Since $U_L(s|G_H, G_L) < v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$ for all $s \in [0, \frac{v_{(n_H)}}{c_H})$, the only $\underline{s}_L$ satisfying the above inequality is $\underline{s}_L = \frac{v_{(n_H)}}{c_H}$, so $\underline{s}_L = \bar{s}_H = \frac{v_{(n_H)}}{c_H}$ and $u_L = v_{(n_H+1)} - v_{(n_H)}\frac{c_L}{c_H}$.

Therefore, over the interval $[0, \frac{v_{(n_H)}}{c_H}]$, the H-cost players' equilibrium strategy must satisfy (10), which gives $G_H(s)$ in (11). Similarly, for $s > \frac{v_{(n_H)}}{c_H}$, the L-cost players' equilibrium strategy must satisfy (12), which gives $G_H(s)$ in (13). ■

Proof of Proposition 4. First, we must have $\bar{s}_L \geq \bar{s}_H$, otherwise there is an aggregate gap $(\bar{s}_L, \bar{s}_H)$. In addition, $\underline{s}_L = 0$ for the same reason.

As a result, if an L-cost player deviates to a performance slightly above 0, he cannot win the top $n_L - 1$ prizes, so his payoff $u_L \leq v_2$.

Since $u_L \leq v_2$ and $\bar{s}_L \geq \bar{s}_H$, we must have $\bar{s}_L \geq (v_{(n)} - v_{(2)})/c_L$. Then, for $s > \bar{s}_H$, the L-cost players' equilibrium strategy $\hat{G}_L(s)$ satisfies

$$\alpha C_2^{n_L-1}\hat{G}_L^2(s) + \Delta_2^1 C_1^{n_L-1}\hat{G}_L(s) + v_{(2)} - c_L s = u_L \tag{27}$$

Recall that $G_L(s)$ given in (14) is a solution to

$$\alpha C_2^{n_L-1}G_L^2(s) + \Delta_2^1 C_1^{n_L-1}G_L(s) + v_{(2)} - c_L s = v_{(2)} \tag{28}$$

Therefore, if $u_L < v_{(2)}$, we have $\hat{G}_L(s) < G_L(s)$ in their common support. Moreover, given others' equilibrium strategies, the H-cost player's payoff by choosing $\bar{s}_H$ is

$$\begin{aligned} & \alpha C_2^{n_L}\hat{G}_L^2(\bar{s}_H) + C_1^{n_L}\hat{G}_L(\bar{s}_H) - c_H\bar{s}_H \\ < \quad & \alpha C_2^{n_L}G_L^2(\bar{s}_H) + C_1^{n_L}G_L(\bar{s}_H) - c_H\bar{s}_H = 0 \end{aligned}$$

which can never happen in an equilibrium. Thus, $u_L = v_{(2)}$ and $\bar{s}_L = (v_{(n)} - v_{(2)})/c_L$.

Substituting $u_L = v_{(2)}$ into (27), it becomes (28). Thus, the L-cost players' strategy $G_L(s)$ for $s > \bar{s}_H$ is given by (14). Moreover, the H-cost player's payoff by choosing $\bar{s}_H$ is

$$\alpha C_2^{n_L}G_L^2(\bar{s}_H) + C_1^{n_L}G_L(\bar{s}_H) - c_H\bar{s}_H = 0$$

whose unique solution is $\bar{s}_H = 0$. Hence, the H-cost player chooses non-performance with certainty, and the L-cost players' strategy is given by (14) for $s \in [0, (v_{(n)} - v_{(2)})/c_L]$. ■

C Omitted Proofs in Section 4

For simpler notation, we use $r = c_L/c_H$. In addition, let $x = c_H s$ and $F_H(x) = G_H(x/c_H)$. Recall that if $n_L = 1$ and $n_H = n - 1$, the H-cost player's equilibrium strategy for $s \in [0, \underline{s}_L]$ is a solution to

$$\alpha C_2^{n-2}G_H^2 + C_1^{n-2}G_H - c_H s = 0$$

and his strategy for $s \in [\underline{s}_L, v_{(n)}/c_H]$ is a solution to

$$\alpha C_2^{n-1}G_H^2 + C_1^{n-1}G_H - c_L s = v_{(n)}(1 - r)$$

Let $\hat{F}(x)$ and $\bar{F}(x)$ be the solution to

$$\begin{aligned} \alpha C_2^{n-2}\hat{F}_H^2(x) + C_1^{n-2}\hat{F}_H(x) &= x \qquad (29) \\ \alpha C_2^{n-1}\bar{F}_H^2(x) + C_1^{n-1}\bar{F}_H(x) &= v_{(n)}(1-r) + rx \qquad (30) \end{aligned}$$

and

$$\begin{aligned} \bar{F}_H(x) &= \frac{-C_1^{n-1} + \sqrt{(C_1^{n-1})^2 + 4\alpha C_2^{n-1}(v_{(n)}(1-r) + rx)}}{2\alpha C_2^{n-1}} \qquad (31) \\ \hat{F}_H(x) &= \frac{-C_1^{n-2} + \sqrt{(C_1^{n-2})^2 + 4\alpha C_2^{n-2}x}}{2\alpha C_2^{n-2}} \qquad (32) \end{aligned}$$

Then, $F_H(x) = \hat{F}_H(x)$ for $x \in [0, \hat{x}]$ and $F_H(x) = \bar{F}_H(x)$ for $x \in [\hat{x}, v_{(n)}]$, where $\hat{x}$ solves $\hat{F}_H(x) = \bar{F}_H(x)$.

Notice that $\bar{F}_H(\hat{x})$ and $\hat{x}$ are solution to the equation system (29) and (30). Multiplying both sides of (29) by $n-1$ and multiplying both sides of (30) by $n-3$, we obtain

$$\begin{aligned} \alpha\frac{(n-1)(n-2)(n-3)}{2}\bar{F}_H^2(\hat{x}) + (n-1)(n-2)\bar{F}_H(\hat{x}) &= (n-1)\hat{x} \\ \alpha\frac{(n-1)(n-2)(n-3)}{2}\bar{F}_H^2(\hat{x}) + (n-1)(n-3)\bar{F}_H(\hat{x}) &= (n-3)(v_{(n)}(1-r) + r\hat{x}) \end{aligned}$$

We can cancel the term with $\bar{F}_H^2(\hat{x})$ and obtain

$$\bar{F}_H(\hat{x}) = \hat{x}\left(1 - \frac{n-3}{n-1}r\right) - \frac{n-3}{n-1}v_{(n)}(1-r)$$

Substituting (31) into the above equation, we obtain

$$\frac{-C_1^{n-1} + \sqrt{(C_1^{n-1})^2 + 4\alpha C_2^{n-1}(v_{(n)}(1-r) + r\hat{x})}}{2\alpha C_2^{n-1}} = \hat{x} - \frac{n-3}{n-1}[v_{(n)} - r(v_{(n)} - \hat{x})] \qquad (33)$$

which can be rewritten as a quadratic equation of $\hat{x}$. Therefore, we can also have a closed-form characterization of $\hat{x}$. Since $\underline{s}_L = \hat{x}/c_H$, so we also have a closed-form characterization of $\underline{s}_L$.

Lemma 1 *Suppose there is one L-cost player and $n-1$ H-cost players.*

1) If $\alpha = 0$, then

$$\mathbb{E}[c_H s_H] = \frac{n-1}{2}\frac{(n-2) - r(n-3)}{(n-1) - r(n-2)} \qquad (34)$$

2) $\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha} < 0$.

Proof. Since $n_L = 1$ and $\alpha = 0$, the L-cost player's strategy satisfies

$$(n-1)G_H - c_L s = u_L$$

whose solution is

$$\bar{G}_H(s) = \frac{u_L + c_L s}{n-1}$$

Similarly, if an H-cost player chooses a performance level close to 0, his payoff is

$$(n-2)G_H - c_H s = 0$$

whose solution is

$$\hat{G}_H(s) = \frac{c_H s}{n-2}$$

Let $\hat{s}_H$ be the solution to $\hat{G}_H(s) = \bar{G}_H(s)$. That is

$$\frac{c_H s}{n-2} = \frac{u_L + c_L s}{n-1}$$

so

$$\hat{s}_H = \frac{v_{(n)}}{n-1} \frac{1 - c_L/c_H}{\frac{c_H}{n-2} - \frac{c_L}{n-1}}$$

and

$$c_H \hat{s}_H = v_{(n)} \frac{1-r}{\frac{n-1}{n-2} - r}$$

Hence,

$$c_H \mathbb{E}[s_H] = c_H \int_0^{\hat{s}_H} s d\hat{G}_H(s) + c_H \int_{\hat{s}_H}^{\bar{s}_H} s d\bar{G}_H(s)$$

where

$$c_H \int_0^{\hat{s}_H} s d\hat{G}_H(s) = \int_0^{\hat{s}_H} c_H s d\frac{c_H s}{n-2} = \frac{v_{(n)}^2}{2\,(n-2)} \left(\frac{1-r}{\frac{n-1}{n-2} - r} \right)^2$$

and

$$\begin{aligned} c_H \int_{\hat{s}_H}^{\bar{s}_H} s d\bar{G}_H(s) &= \int_{\hat{s}_H}^{v_{(n)}/c_H} \frac{c_L}{n-1} c_H s ds = \int_{\hat{s}_H}^{v_{(n)}/c_H} \frac{1}{n-1} \frac{c_L}{c_H} c_H s d c_H s \\ &= \int_{c_H \hat{s}_H}^{v_{(n)}} \frac{r}{n-1} t dt = \frac{r v_{(n)}^2}{2(n-1)} \left[1 - \left(\frac{1-r}{\frac{n-1}{n-2} - r} \right)^2 \right] \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[c_H s_H] &= \frac{v_{(n)}^2}{2} \left[\frac{1}{n-2} \left(\frac{1-r}{\frac{n-1}{n-2} - r} \right)^2 + \frac{r}{n-1} \left(1 - \left(\frac{1-r}{\frac{n-1}{n-2} - r} \right)^2 \right) \right] \\ &= \frac{n-1}{2} \frac{(n-2) - r(n-3)}{(n-1) - r(n-2)} \end{aligned}$$

where the second equality is from $v_{(n)} = n-1$ when $\alpha = 0$.

Next, we prove $\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha} < 0$ in three steps. First, $F_H(x)$ is decreasing in α. If α increases, (29) implies that $\hat{F}_H(x)$ decreases, and (30) implies $\bar{F}_H(x)$ decreases. Therefore, $F_H(x)$ decreases as α increases.

Second, $\frac{\partial F_H(x)}{\partial r}$ is increasing in α. Equation (29) implies that $\frac{\partial \hat{F}_H(x)}{\partial r}$ is independent of α. Taking derivatives of both sides in (30) w.r.t. r, we get

$$[\alpha \bar{F}_H(x)(n-1)(n-2)+(n-1)]\frac{\partial \bar{F}_H(x)}{\partial r} = -(v_{(n)}-x) \tag{35}$$

If α increases, the first step implies that $\bar{F}_H(x)$ decreases, so $\frac{\partial \bar{F}_H(x)}{\partial r}$ has to increase according to (35). Hence, $\frac{\partial F_H(x)}{\partial r}$ increases as α increases.

Third, $\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha} < 0$. To see this, notice that

$$\begin{aligned} \frac{\partial \mathbb{E}[c_H s_H]}{\partial r} &= \frac{\partial}{\partial r}\left(v_{(n)} F_H(v_{(n)}) - \int_0^{v_{(n)}} F_H(x)dx \right) \\ &= -\int_0^{v_{(n)}} \frac{\partial F_H(x)}{\partial r} dx \end{aligned}$$

which decreases if α increases because of the second step. ■

Proof of Proposition 5. Propositions 1-4 characterizes the boundaries of equilibrium strategies' supports in closed form, so it is straightforward to verify that $\underline{s}_H^{\text{before}} = \underline{s}_H^{\text{after}} \leq \bar{s}_H^{\text{after}} \leq \bar{s}_H^{\text{before}}$ and $\underline{s}_L^{\text{after}} \leq \underline{s}_L^{\text{before}} < \bar{s}_L^{\text{before}} \leq \bar{s}_L^{\text{after}}$.

In the next three paragraphs, we show $G_H^{\text{after}}(s) \geq G_H^{\text{before}}(s)$ for s in their common support. We start with the case $n_L = 0$, so there are n H-cost players in the contest. The equilibrium G_H^{before} is characterized in Proposition 1, and it satisfies

$$\alpha C_2^{n-1}(G_H^{\text{before}})^2 + C_1^{n-1} G_H^{\text{before}} - c_H s = 0 \tag{36}$$

When there is one more L-cost player, we have $n_L = 1$ and the equilibrium G_H^{after} is characterized in Proposition 2, and it satisfies

$$\alpha C_2^{n-2}(G_H^{\text{after}})^2 + C_1^{n-2} G_H^{\text{after}} - c_H s = 0 \tag{37}$$

for $s \in [0, \underline{s}_L]$, and

$$\alpha C_2^{n-1}(G_H^{\text{after}})^2 + C_1^{n-1} G_H^{\text{after}} - c_L s = u_L^{\text{after}} \tag{38}$$

for $s \in [\underline{s}_L, v_{(n)}/c_H]$. Comparing (36) and (37), we obtain $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for $s \in [0, \underline{s}_L]$. Notice that $u_L^{\text{after}} + c_L s - c_H s$ is decreasing in s, so $u_L^{\text{after}} + c_L s - c_H s \geq u_L^{\text{after}} + (c_L - c_H) v_{(n)}/c_H = 0$ for $s \leq v_{(n)}/c_H$. Thus, comparing (36) and (38), we have $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for $s \in [\underline{s}_L, v_{(n)}/c_H]$.

Next, suppose $n_L = 1$. Then, the equilibrium is characterized in Proposition 2. Similar to (37) and (38), the equilibrium strategy G_H^{before} satisfies

$$\alpha C_2^{n-2}(G_H^{\text{before}})^2 + C_1^{n-2} G_H^{\text{before}} - c_H s = 0 \tag{39}$$

for $s \in [0, \underline{s}_L]$, and

$$\alpha C_2^{n-1}(G_H^{\text{before}})^2 + C_1^{n-1} G_H^{\text{before}} - c_L s = u_L^{\text{after}} \tag{40}$$

for $s \in [\underline{s}_L, v_{(n)}/c_H]$. When there is one more L-cost player, we have $n_L = 2$. Moreover, if $n = 3$, Proposition 4 implies the H-cost player chooses non-performance with certainty. Thus, $G_H^{\text{after}}(0) = 1$, and therefore $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$. If $n > 3$ and $n_L = 2$, the equilibrium is characterized in Proposition 3, and the strategy G_H^{after} satisfies

$$\alpha C_2^{n-3}(G_H^{\text{after}})^2 + C_1^{n-3} G_H^{\text{after}} - c_H s = 0 \tag{41}$$

Comparing the above equation with (40), we have $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for $s \in [0, \frac{v_{(n_H)}}{c_H}] \cap [0, \underline{s}_L]$. If $\frac{v_{(n_H)}}{c_H} \leq \underline{s}_L$, we already prove $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for their common support. If $\frac{v_{(n_H)}}{c_H} > \underline{s}_L$, then consider $s \in [\underline{s}_L, \frac{v_{(n_H)}}{c_H}]$. In this interval, we have $u_L^{\text{after}} + c_L s - c_H s \geq u_L^{\text{after}} + (c_L - c_H)\frac{v_{(n_H)}}{c_H} = v_{(n)}(1 - c_L/c_H) + (c_L - c_H)\frac{v_{(n_H)}}{c_H} > 0$. Therefore, comparison of (40) and (41) implies $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for $s \in [\underline{s}_L, \frac{v_{(n_H)}}{c_H}]$. Thus, $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ over their common support.

Consider $n_L = 2$. We already discussed $n = 3$ above, so suppose $n \geq 4$. If $n = 4$, then if there is one more L-cost player, we have $n_L = 3$ and $n_H = 1$. Therefore, the H-cost player chooses non-performance with certainty, and therefore $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ over their common support. Consider the case $n > 4$. Then, the equilibrium is characterized by Proposition 3, and the equilibrium strategy G_H^{before} satisfies

$$\alpha C_2^{n-3}(G_H^{\text{before}})^2 + C_1^{n-3} G_H^{\text{before}} - c_H s = 0 \tag{42}$$

If there is one more L-cost player, the equilibrium strategy G_H^{after} satisfies

$$\alpha C_2^{n-4}(G_H^{\text{after}})^2 + C_1^{n-4} G_H^{\text{after}} - c_H s = 0 \tag{43}$$

Comparing (42) and (43), we have $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$. We can continue in the same way to verify $G_H^{\text{before}}(s) \leq G_H^{\text{after}}(s)$ for larger $n_L = 3, ..., n-2$.

In the remainder of the proof, we compare $\mathbb{E}[s_L^{\text{after}}]$ and $\mathbb{E}[s_L^{\text{before}}]$. First, consider $n_L = 1$. Recall that the expected winnings of an H-cost player is $\mathbb{E}[c_H s_H]$, so the expected winnings of the H-cost player is $V - (n-1)\mathbb{E}[c_H s_H]$. The L-cost player's expected performance is

$$\mathbb{E}[s_L^{\text{before}}] = \frac{W_L - u_L}{c_L} = \frac{V - (n-1)\mathbb{E}[c_H s_H] - v_{(n)}(1-r)}{c_L}$$

If there is one more L-cost player, then

$$\mathbb{E}[s_L^{\text{after}}] = \frac{v_{(n)} + v_{(n-1)}}{2c_L} - \frac{v_{(n-1)}}{c_L} + \frac{v_{(n-2)}}{c_H}$$

Therefore,

$$\begin{aligned}\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]} &= \frac{V-(n-1)\mathbb{E}[c_H s_H]-v_{(n)}(1-r)}{\frac{v_{(n)}-v_{(n-1)}}{2}+v_{(n-2)}r}\\ &= \frac{V-v_{(n)}-(n-1)\mathbb{E}[c_H s_H]+v_{(n)}r}{\frac{v_{(n)}-v_{(n-1)}}{2}+v_{(n-2)}r}\end{aligned}$$

In the remainder of this paragraph, we show that $\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}<1$ if and only if r is sufficiently small. Moreover, it is sufficient to show that $\frac{\partial}{\partial r}\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}>0$ if $\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}=1$. With $\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}=1$, it is straightforward to verify that $\frac{\partial}{\partial r}\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}$ has the same sign with

$$-(n-1)\frac{\partial\mathbb{E}[c_H s_H]}{\partial r}+v_{(n)}-v_{(n-2)}\equiv M(\alpha) \tag{44}$$

From Lemma 1, we have $\frac{\partial^2\mathbb{E}[c_H s_H]}{\partial r\partial\alpha}<0$, so

$$M'(\alpha)=-(n-1)\underbrace{\frac{\partial^2\mathbb{E}[c_H s_H]}{\partial r\partial\alpha}}_{<0}+\underbrace{\frac{\partial(v_{(n)}-v_{(n-2)})}{\partial\alpha}}_{>0}>0$$

Thus, it remains to show $M(\alpha)>0$ for $\alpha=0$. Lemma 1 implies that $\mathbb{E}[c_H s_H]=\frac{n-1}{2}\frac{(n-2)-r(n-3)}{(n-1)-r(n-2)}$, so we can rewrite (44) as

$$M(0)=-\frac{1}{2\left(1-r\frac{n-2}{n-1}\right)^2}+2>-\frac{1}{2\left(1-\frac{1}{2}\right)^2}+2=0$$

Hence, $M(0)>0$ and $M'(\alpha)>0$ imply $M(\alpha)>0$, which means $\frac{\partial}{\partial r}\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}>0$ if $\frac{\mathbb{E}[s_L^{\text{before}}]}{\mathbb{E}[s_L^{\text{after}}]}=1$.

Next, consider $n_L=2,...,n-2$. Then, we have either the equilibrium with separation or the equilibrium with non-performance. In either case, the total prize the L-cost players win is $\sum_{k=n_H+1}^n v_{(k)}$ and their equilibrium payoffs are $u_L=v_{(n_H+1)}-c_L v_{(n_H)}/c_H$. Therefore, we have $\sum_{k=n_H+1}^n v_{(k)}-n_L c_L\mathbb{E}[s_L]=n_L u_L$, where the mean $\mathbb{E}[s_L]$ is calculated using distribution G_L. Thus,

$$\mathbb{E}[s_L]=\left(\frac{\sum_{k=n_H+1}^n v_{(k)}}{n_L}-u_L\right)\frac{1}{c_L}$$

Recall that $v_{(k)}$ is weakly convex in k so the average value of the top n_L prizes: $\frac{\sum_{k=n_H+1}^n v_{(k)}}{n_L}$ decreases in n_L. Moreover, we can verify that $u_L=v_{(n_H+1)}-c_L v_{(n_H)}/c_H$ is decreasing in $n_L=n-n_H$, which means if there are more L-cost players, they earn lower equilibrium payoffs. Therefore, it is not obvious whether $\mathbb{E}[s_L]$ increases with n_L. However, we show below that $\mathbb{E}[s_L]$ is indeed increasing in n_L. Substituting $u_L=v_{(n_H+1)}-c_L v_{(n_H)}/c_H$ and $n_H=n-n_L$ into the expression of $\mathbb{E}[s_L]$, we obtain

$$\mathbb{E}[s_L] = \frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{c_L n_L} - \frac{v_{(n-n_L+1)}}{c_L} + \frac{v_{(n-n_L)}}{c_H}$$

Let $\mathbb{E}[s_L^{\text{before}}]$ be the above expected performance. If there is one more L-cost player, an L-cost player's expected equilibrium performance is

$$\mathbb{E}[s_L^{\text{after}}] = \frac{\sum_{k=n-n_L}^{n} v_{(k)}}{c_L(n_L+1)} - \frac{v_{(n-n_L)}}{c_L} + \frac{v_{(n-n_L-1)}}{c_H}$$

Recall that $c_H > 2c_L$, so to show $\mathbb{E}[s_L^{\text{after}}] > \mathbb{E}[s_L^{\text{before}}]$, it is sufficient to show

$$\frac{\sum_{k=n-n_L}^{n} v_{(k)}}{n_L+1} - v_{(n-n_L)} + \frac{c_L}{c_H} v_{(n-n_L-1)} > \frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{n_L} - v_{(n-n_L+1)} + \frac{c_L}{c_H} v_{(n-n_L)}$$

which can be rewritten as

$$\alpha + (1-r)\left(v_{(n-n_L)} - v_{(n-n_L-1)}\right) - \left(\frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{n_L} - \frac{\sum_{k=n-n_L}^{n} v_{(k)}}{n_L+1}\right) > 0$$

Let the left hand side be $L(\alpha)$. We can verify that $L(0) = \frac{1}{2} - r > 0$. Moreover, using (15) and (19), we have

$$\begin{aligned} L'(\alpha) &= 1 + (1-r)(n-n_L-2) - \frac{(2n-n_L-1)(n-n_L-2)}{6} \\ &\quad + \frac{(2n-n_L-2)(n-n_L-3)}{6} \\ &= \frac{4-n_L}{6} + \left(\frac{1}{2} - r\right)(n-n_L-2) \end{aligned}$$

Since $L(\alpha)$ is a linear function, and recall that $L(\alpha) = \mathbb{E}[s_L^{\text{after}}] - \mathbb{E}[s_L^{\text{before}}]$, so we have

$$\mathbb{E}[s_L^{\text{after}}] - \mathbb{E}[s_L^{\text{before}}] = \frac{1}{2} - r + \alpha\left(\frac{4-n_L}{6} + \left(\frac{1}{2} - r\right)(n-n_L-2)\right)$$

If $\alpha \to 0$, we have $\mathbb{E}[s_L^{\text{after}}] - \mathbb{E}[s_L^{\text{before}}]$ converges to $\frac{1}{2} - r > 0$. If $n_L \le 4$, we always have $\mathbb{E}[s_L^{\text{after}}] > \mathbb{E}[s_L^{\text{before}}]$. If $4 < n_L < n-1$, then $\mathbb{E}[s_L^{\text{after}}] > \mathbb{E}[s_L^{\text{before}}]$ if and only if

$$\frac{1}{2} - r + \alpha\left(\frac{4-n_L}{6} + \left(\frac{1}{2} - r\right)(n-n_L-2)\right) > 0 \tag{45}$$

Finally, consider $n_L = n-1$. Then,

$$\begin{aligned} \mathbb{E}[s_L^{\text{before}}] &= \frac{\sum_{k=2}^{n} v_{(k)}}{c_L(n-1)} - \frac{1}{c_L} \\ \mathbb{E}[s_L^{\text{after}}] &= \frac{\sum_{k=1}^{n} v_{(k)}}{c_L n} \end{aligned}$$

	$n_L = 0$	$n_L = 1$	$2 \le n_L \le n-2$	$n_L = n-1$
$n = 3$	Lemma 2	Lemma 4 if $\alpha = 0$ Lemma 6 if $\alpha > 0$	Lemma 3	
$n \ge 4$		Lemma 5 if $\alpha = 0$ Lemma 7 if $\alpha > 0$	Lemma 8	Lemma 3

Table 2: Lemmas for Proposition 6

Recall that $v_{(1)} = 0$, so $\sum_{k=2}^{n} v_{(k)} = \sum_{k=1}^{n} v_{(k)} = V$. Then, using (16), we have

$$c_L(\mathbb{E}[s_L^{\text{after}}] - \mathbb{E}[s_L^{\text{before}}]) = 1 - \frac{C_2^n + \alpha C_3^n}{n(n-1)}$$

which is linear function of α with a negative slope. Therefore, $\mathbb{E}[s_L^{\text{after}}] < \mathbb{E}[s_L^{\text{before}}]$ if and only if $\alpha > C_2^n / C_3^n = \frac{3}{n-2}$.

Next, we characterize $\gamma_\alpha(n_L/n)$. Suppose that the prize sequence is sufficiently convex so that $\alpha > \frac{3}{n-2}$. Then, according to the first step above, $\mathbb{E}[s_L^{\text{after}}] > \mathbb{E}[s_L^{\text{before}}]$ if $n_L = 1$. Thus, $\gamma_\alpha(\frac{1}{n}) = \frac{1}{2}$. Moreover, the third step above implies that for those α values, we have $\mathbb{E}[s_L^{\text{after}}] < \mathbb{E}[s_L^{\text{before}}]$ if $n_L = n-1$. Thus, $\gamma_\alpha(\frac{n-1}{n}) = 0$. In the second step above, $\mathbb{E}[s_L^{\text{after}}] > \mathbb{E}[s_L^{\text{before}}]$ if and only if (45), which can be rewritten as

$$r < \frac{1}{2} - \frac{\alpha(\frac{n_L}{n} - \frac{4}{n})}{\frac{6}{n} + 6\alpha(1 - \frac{n_L}{n} - \frac{2}{n})} \tag{46}$$

Notice that for $n_L \le 4$, the right hand side of (46) is above $\frac{1}{2}$, so the inequality holds for all r. Thus, $\gamma_\alpha(\frac{n_L}{n}) = \frac{1}{2}$ for $\frac{n_L}{n} \le \frac{4}{n}$. For $\frac{n_L}{n} > \frac{4}{n}$, the right hand side of (46) is below $\frac{1}{2}$. Moreover, it is decreasing in $\frac{n_L}{n}$, so its minimum is reached at $\frac{n_L}{n} = \frac{n-2}{n}$, and the minimum is $\frac{1}{2} - \frac{\alpha(n_L-4)}{6}$. Thus, for $\frac{n_L}{n} > \frac{4}{n}$, we have $\gamma_\alpha(\frac{n_L}{n}) = \max\{\frac{1}{2} - \frac{\alpha(n_L-4)}{6}, 0\}$. In summary, if $\alpha > \frac{3}{n-2}$, we have a weakly decreasing function

$$\gamma_\alpha\left(\frac{n_L}{n}\right) = \begin{cases} \frac{1}{2} & \text{if } \frac{n_L}{n} \le \frac{4}{n} \text{ and } \frac{n_L}{n} < \frac{n-1}{n} \\ \max\{\frac{1}{2} - \frac{\alpha n}{6}(\frac{n_L}{n} - \frac{4}{n}), 0\} & \text{if } \frac{4}{n} < \frac{n_L}{n} < \frac{n-1}{n} \\ 0 & \text{if } \frac{n_L}{n} = \frac{n-1}{n} \end{cases}$$

■

We prove Proposition 6 through Lemmas 2 to 8. The case covered by each lemma is summarized in Table 2. In particular, Lemma 2 prove the proposition for $n_L = 0$, Lemma 3 proves the proposition for $n_L = n-1$, and Lemmas 4 to 8 show the proposition for $n_L = 2, ..., n-2$. We use $\Pi_{\text{other}}^{\text{after}}(n_L)$ for the total expected equilibrium performance of the top $n-1$ players in a contest with n_L L-cost players. After the entry of an L-cost player, there are $n_L + 1$ L-cost players, and we use $\Pi_{\text{other}}^{\text{after}}(n_L)$ for the total expected equilibrium performance of the bottom $n-1$ players in the contest.

Lemma 2 *Consider a contest with n H-cost players. If an L-cost player enters the contest,*

$\Pi^{after}_{other}(0) < \Pi^{before}_{other}(0)$ *for all* $\alpha \geq 0$.

Proof. Before the entry, n H-cost players compete in the contest. In the equilibrium, each player's expected winnings is V/n, and his expected payoff is 0. Therefore, an H-cost player's expected performance is $V/(nc_H)$ and the total expected performance of the $n-1$ H-cost players is $\Pi^{\text{before}}_{\text{other}}(0) = \frac{n-1}{nc_H}V$.

After the L-cost player enters, he competes with $n-1$ H-cost players. Recall that in the proof of Proposition 2, we show that $G_H(s) \geq G_L(s)$ over their common support. As a result, the L-cost player's expected winnings W_L is no lower than that of an H-cost player, W_H. Because the total winnings equals the total prize, we have $W_L + (n-1)W_H = V$. Therefore, $W_L \geq W_H$ implies $W_H \leq V/n$. An H-cost player's expected performance is $W_H/c_H < V/(nc_H)$. Hence, the $n-1$ H-cost players' total expected performance is $\Pi^{\text{after}}_{\text{other}}(0) = \frac{(n-1)W_H}{c_H} < \frac{n-1}{nc_H}V = \Pi^{\text{before}}_{\text{other}}(0)$. ■

The following lemma compares the other players' performance before and after the entry of an L-cost player if $n_L = n-1$. It shows that $\phi_\alpha(\frac{n_L+1}{n}) = 0$ or $1/2$. Hence, it remains to show Proposition 6 for $n_L = 2, ..., n-2$.

Lemma 3 *Suppose* $n_L = n-1$. *Then, there exists* $\hat{\alpha} > 0$ *such that* $\Pi^{after}_{other}(n-1) > \Pi^{before}_{other}(n-1)$ *if* $\alpha < \hat{\alpha}$ *and* $\Pi^{after}_{other}(n-1) < \Pi^{before}_{other}(n-1)$ *if* $\alpha > \hat{\alpha}$.

Proof. Proposition 4 implies that if $n_L = n-1$, the H-cost player chooses non-performance with certainty in the equilibrium. Therefore, the $n-1$ L-cost players compete for $v_{(2)}, ..., v_{(n)}$, and their payoffs are $v_{(2)} = 1$. Therefore, the total expected performance of $n-1$ L-cost players is $\Pi^{\text{before}}_{\text{other}}(n-1) = (V-(n-1))/c_L$.

After the entry of another L-cost player, the equilibrium payoff of an L -cost player becomes 0. Because all the prizes are won by n L-cost players, each has expected winnings V/n. Therefore, an L-cost player's expected performance is $V/(c_L n)$, and the total performance of $n-1$ L-cost players is $\Pi^{\text{after}}_{\text{other}}(n-1) = (\frac{n-1}{n}V)/c_L$. Hence, $\Pi^{\text{before}}_{\text{other}}(n-1) - \Pi^{\text{after}}_{\text{other}}(n-1) = \frac{1}{c_L}[V/n-(n-1)]$. If $\alpha = 0$, then $V = 1+2+...+(n-1) = \frac{n(n-1)}{n}$, so $\Pi^{\text{before}}_{\text{other}}(n-1) - \Pi^{\text{after}}_{\text{other}}(n-1) < 0$. If α increases, V increases and $\Pi^{\text{before}}_{\text{other}}(n-1) - \Pi^{\text{after}}_{\text{other}}(n-1)$ increases. As a result, there exists $\hat{\alpha} > 0$ such that $\Pi^{\text{before}}_{\text{other}}(n-1) - \Pi^{\text{after}}_{\text{other}}(n-1) > 0$ if and only if $\alpha > \hat{\alpha}$. ■

The following two lemmas consider the case with $\alpha = 0$. In particular, Lemma 4 considers the case with $n = 3$, and Lemma 5 considers $n \geq 4$.

Lemma 4 *If* $n = 3$ *and* $\alpha = 0$, *then* $\Pi^{after}_{other}(1)/\Pi^{before}_{other}(1)$ *is decreasing in* c_L/c_H.

Proof. If $n_L = 2$, the H-cost player chooses non-performance with probability 1, so the two L-cost players compete for the top two prizes. Therefore, $\Pi^{\text{after}}_{\text{other}}(1) = \mathbb{E}[s^{\text{after}}_L] = \left(\frac{v_{(3)}+v_{(2)}}{2} - u^{\text{after}}_L\right)/c_L = \frac{v_{(3)}-v_{(2)}}{2c_L}$, where the last equality is from $u^{\text{after}}_L = 1$.

If $n_L = 1$, we have $\Pi^{\text{before}}_{\text{other}}(1) = \mathbb{E}[s_L] + (n-2)\mathbb{E}[s_H]$. Notice that the expected winnings of an H-cost player is $c_H\mathbb{E}[s_H]$, so the total winnings of the H-cost players is $(n-1)c_H\mathbb{E}[s_H]$.

Therefore, the total expected winnings of the L-cost player is $W_L = V - (n-1)c_H\mathbb{E}[s_H]$. Notice that $u_L = v_{(n)}(1-r)$, so

$$\mathbb{E}[s_L] = \frac{W_L - u_L}{c_L} = \frac{V - (n-1)c_H\mathbb{E}[s_H] - v_{(n)}(1-r)}{c_L}$$

Substituting $\mathbb{E}[s_L]$ into $\Pi^{\text{before}}_{\text{other}}(1)$, we obtain

$$\begin{aligned}
\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} &= \frac{(v_{(3)} - v_{(2)})/(2c_L)}{\frac{V-(n-1)c_H\mathbb{E}[s_H]-v_{(n)}(1-r)}{c_L} + (n-2)\mathbb{E}[s_H]} \\
&= \frac{(v_{(3)} - v_{(2)})/2}{V - v_{(n)}(1-r) - ((n-1) - r(n-2))\,\mathbb{E}[c_H s_H]} \qquad (47)\\
&= \frac{(1+\alpha)/2}{3+\alpha - (2+\alpha)(1-r) - (2-r)\,\mathbb{E}[c_H s_H]} \qquad (48)
\end{aligned}$$

Substituting $\alpha = 0$ and $\mathbb{E}[c_H s_H]$ in (34) into (48), we obtain

$$\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} = \frac{1/2}{3 - 2(1-r) - 1} = \frac{1}{4r}$$

which is decreasing in r. ■

Lemma 5 *If $n \geq 4$ and $\alpha = 0$, $\Pi^{\text{after}}_{\text{other}}(1)/\Pi^{\text{before}}_{\text{other}}(1)$ is decreasing in c_L/c_H.*

Proof. Following the same reasoning in the beginning of Lemma 4's proof, we have

$$\Pi^{\text{after}}_{\text{other}}(1) = \mathbb{E}[s_L] + (n-2)\mathbb{E}[s_H] = \frac{v_{(n)} - v_{(n-1)} + 2v_{(n-2)}r}{2c_L} + \frac{\sum_{k=1}^{n-2} v_{(k)}}{c_H}$$

and

$$\begin{aligned}
\Pi^{\text{before}}_{\text{other}}(1) &= \mathbb{E}[s_L] + (n-2)\mathbb{E}[s_H] \\
&= \frac{V - (n-1)c_H\mathbb{E}[s_H] - v_{(n)}(1-r)}{c_L} + (n-2)\mathbb{E}[s_H]
\end{aligned}$$

Therefore,

$$\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} = \frac{(v_{(n)} - v_{(n-1)})/2 + (v_{(n-2)} + \sum_{k=1}^{n-2} v_{(k)})r}{V - v_{(n)}(1-r) - ((n-1) - r(n-2))\,\mathbb{E}[c_H s_H]} \qquad (49)$$

Substituting $\alpha = 0$ and $\mathbb{E}[c_H s_H]$ in (34) into (49), we have

$$\begin{aligned}
\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} &= \frac{(v_{(n)} - v_{(n-1)})/2 + (v_{(n-2)} + \sum_{k=1}^{n-2} v_{(k)})r}{V - v_{(n)}(1-r) - \frac{v_{(n)}^2}{2}\frac{n-2-r(n-3)}{n-1}} \\
&= \frac{1/2 + \left(n-3+\frac{(n-3)(n-2)}{2}\right)r}{\frac{(n-1)n}{2} - (n-1)(1-r) - \frac{(n-1)^2}{2}\frac{n-2-r(n-3)}{n-1}} \\
&= \frac{\frac{1}{r} + n(n-3)}{(n-1)^2} \qquad (50)
\end{aligned}$$

where the second equality comes from $\alpha = 0$. Therefore, $\Pi^{\text{after}}_{\text{other}}(1)/\Pi^{\text{before}}_{\text{other}}(1)$ is decreasing in r. ■

Given that we know the comparison of $\Pi^{\text{after}}_{\text{other}}(1)$ and $\Pi^{\text{before}}_{\text{other}}(1)$ if $\alpha = 0$, let us consider how the comparison changes if α increases from 0. In particular, Lemma 6 considers $n = 3$ and Lemma 7 considers $n \geq 4$.

Lemma 6 *If $n = 3$, $\Pi^{after}_{other}(1)/\Pi^{before}_{other}(1)$ is decreasing in c_L/c_H if $\alpha > 0$.*

Proof. According to (47), it is sufficient to show $A \equiv [(n-1)-(n-2)r]\mathbb{E}[c_H s_H] > 0$ is decreasing in r. By the definition of A, we have

$$\frac{\partial A}{\partial r} = -(n-2)\mathbb{E}[c_H s_H] + [(n-1)-(n-2)r]\frac{\partial \mathbb{E}[c_H s_H]}{\partial r} \tag{51}$$

On the one hand, if α increases, the first step in the proof of Lemma 1 implies that $\mathbb{E}[c_H s_H] = \int_0^{v_{(n)}} x dF_H(x)$ increases. On the other hand, Lemma 1 implies that $\frac{\partial \mathbb{E}[c_H s_H]}{\partial r}$ decreases if α increases. Then, (51) implies that $\frac{\partial A}{\partial r}$ decreases if α increases. Next we show $\frac{\partial A}{\partial r} < 0$. If $\alpha = 0$, the proofs of Lemmas 4 and 5 imply that $\frac{\partial A}{\partial r} < 0$. As α increases, $\frac{\partial A}{\partial r}$ decreases as shown above. Therefore, $\frac{\partial A}{\partial r} < 0$ for all $\alpha \geq 0$. ■

Lemma 7 *Suppose $n \geq 4$ and $\alpha > 0$. Then, $\frac{\partial}{\partial r}\left(\frac{\Pi^{after}_{other}(1)}{\Pi^{before}_{other}(1)}\right) < 0$ wherever $\frac{\Pi^{after}_{other}(1)}{\Pi^{before}_{other}(1)} = 1$.*

Proof. Recall that (49) implies

$$\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} = \frac{(v_{(n)} - v_{(n-1)})/2 + \left(v_{(n-2)} + \sum_{k=1}^{n-2} v_{(k)}\right) r}{\sum_{k=1}^{n-1} v_{(k)} + v_{(n)} r - ((n-1) - r(n-2))\mathbb{E}[c_H s_H]}$$

so

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}\right) &= \frac{\frac{\partial \Pi^{\text{after}}_{\text{other}}(1)}{\partial r}\Pi^{\text{before}}_{\text{other}}(1) - \Pi^{\text{after}}_{\text{other}}(1)\frac{\partial \Pi^{\text{before}}_{\text{other}}(1)}{\partial r}}{\left(\Pi^{\text{before}}_{\text{other}}(1)\right)^2} \\
&= \frac{\left(v_{(n-2)} + \sum_{k=1}^{n-2} v_{(k)}\right)\Pi^{\text{before}}_{\text{other}}(1) - \Pi^{\text{after}}_{\text{other}}(1)\frac{\partial \Pi^{\text{before}}_{\text{other}}(1)}{\partial r}}{\left(\Pi^{\text{before}}_{\text{other}}(1)\right)^2}
\end{aligned}$$

If $\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} = 1$, we have $\Pi^{\text{after}}_{\text{other}}(1) = \Pi^{\text{before}}_{\text{other}}(1) > 0$, so the above expression has the same sign with $\frac{\partial \Pi^{\text{after}}_{\text{other}}(1)}{\partial r} - \frac{\partial \Pi^{\text{before}}_{\text{other}}(1)}{\partial r}$. Therefore, it is sufficient to show $\frac{\partial \Pi^{\text{after}}_{\text{other}}(1)}{\partial r} - \frac{\partial \Pi^{\text{before}}_{\text{other}}(1)}{\partial r} < 0$, or equivalently

$$v_{(n-2)} + \sum_{k=1}^{n-2} v_{(k)} - v_{(n)} - (n-2)\mathbb{E}[c_H s_H] + ((n-1)-(n-2)r)\frac{\partial \mathbb{E}[c_H s_H]}{\partial r} < 0 \tag{52}$$

Let $L(\alpha)$ be the left hand side of (52) as a function of α. According to Lemma 5, we must have

$L(0) < 0$, so it remains to show that $L'(\alpha) < 0$ for $\alpha > 0$.

$$L'(\alpha) = C_2^{n-3} + C_3^{n-2} - C_2^{n-1} + \frac{\partial^2}{\partial r \partial \alpha} (((n-1) - (n-2)r)\mathbb{E}[c_H s_H])$$

where $C_k^m = 0$ if $m < k$.

If $n = 4$, 5 or 6, we have $C_2^{n-3} + C_3^{n-2} - C_2^{n-1} < 0$. Moreover, according to Lemma 1, we have $\frac{\partial^2 A}{\partial r \partial \alpha} < 0$, so $L'(\alpha) = C_2^{n-3} + C_3^{n-2} - C_2^{n-1} + \frac{\partial^2 A}{\partial r \partial \alpha} < 0$.

If $n \geq 7$, we have

$$\begin{aligned}
L'(\alpha) &= \underbrace{C_2^{n-3} + C_3^{n-2} - C_2^{n-1}}_{>0} + \underbrace{\frac{\partial^2 A}{\partial r \partial \alpha}}_{<0} \\
&= \underbrace{C_2^{n-3} + C_3^{n-2} - C_2^{n-1}}_{>0} - (n-2) \underbrace{\frac{\partial \mathbb{E}[c_H s_H]}{\partial \alpha}}_{>0} + ((n-1) - (n-2)r) \underbrace{\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha}}_{<0} \\
&= \underbrace{C_2^{n-3} + C_3^{n-2} - C_2^{n-1} - (n-2)C_2^{n-1}}_{<0} + (n-2) \int_0^{v_{(n)}} \underbrace{\frac{\partial F_H(x)}{\partial \alpha}}_{<0} dx \\
&\quad + ((n-1) - (n-2)r) \underbrace{\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha}}_{<0} \\
&< 0
\end{aligned}$$

where the third equality is from integration by parts, and $\frac{\partial F_H(x)}{\partial \alpha} < 0$ and $\frac{\partial^2 \mathbb{E}[c_H s_H]}{\partial r \partial \alpha} < 0$ are shown in the first and third steps in the proof of Lemma 1. Therefore, $L'(\alpha) < 0$ for $n \geq 4$ and $\alpha > 0$, which completes the proof. ■

The lemma below shows that $\phi_\alpha(n_L/n)$ is unique for $n_H = 2, ..., n-2$.

Lemma 8 *$\Pi_{other}^{after}(n_L)/\Pi_{other}^{before}(n_L)$ is decreasing in c_L/c_H for $n_L = 2, ..., n-2$.*

Proof. For $n_L = 2, ..., n-2$, the contest has an equilibrium of separation, which is characterized in Proposition 3. Thus, before the entry, the total expected winnings of the L-cost players is $W_L = \sum_{k=n-n_L+1}^{n} v_{(k)}$, and the total expected winnings of the H-cost players is $W_H = \sum_{k=1}^{n_H} v_{(k)}$. After the entry, one more prize is won by the L-cost players, and the total expected winnings become $W'_L = \sum_{k=n-n_L}^{n} v_{(k)}$ and $W'_H = \sum_{k=1}^{n_H-1} v_{(k)}$.

Before the entry, the total expected performance of the strongest $n-1$ players is

$$\begin{aligned}
\Pi_{\text{other}}^{\text{before}}(n_L) &= (W_L - n_L u_L)\frac{1}{c_L} + W_H \frac{n_H - 1}{n_H} \frac{1}{c_H} \\
&= \frac{W_L - n_L v_{(n_H+1)}}{c_L} + \frac{n_L v_{(n_H)} + W_H \frac{n_H - 1}{n_H}}{c_H}
\end{aligned}$$

where the second equality is from $u_L = v_{(n_H+1)} - v_{(n_H)} r$. Similarly, after the entry, the total expected performance of the weakest $n-1$ players is

$$\Pi_{\text{other}}^{\text{after}}(n_L) = \frac{W'_L \frac{n_L}{n_L+1} - n_L v_{(n_H)}}{c_L} + \frac{n_L v_{(n_H-1)} + W'_H}{c_H}$$

Thus,

$$\begin{aligned}
\frac{\Pi_{\text{other}}^{\text{after}}(n_L)}{\Pi_{\text{other}}^{\text{before}}(n_L)} &= \frac{(W'_L \frac{n_L}{n_L+1} - n_L v_{(n-n_L)}) + (n_L v_{(n-n_L-1)} + W'_H)r}{(W_L - n_L v_{(n-n_L+1)}) + (n_L v_{(n-n_L)} + W_H \frac{n_H-1}{n_H})r} \\
&= \frac{\frac{\sum_{k=n-n_L}^{n} v_{(k)}}{n_L+1} - v_{(n-n_L)} + \left(v_{(n-n_L-1)} + \frac{\sum_{k=1}^{n_H-1} v_{(k)}}{n_H-1}\frac{n_H-1}{n_L}\right)r}{\frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{n_L} - v_{(n-n_L+1)} + \left(v_{(n-n_L)} + \frac{\sum_{k=1}^{n_H} v_{(k)}}{n_H}\frac{n_H-1}{n_L}\right)r} \qquad (53)
\end{aligned}$$

Denote $C'_L = \frac{\sum_{k=n-n_L}^{n} v_{(k)}}{n_L+1} - v_{(n-n_L)}$, $C'_H = v_{(n-n_L-1)} + \frac{\sum_{k=1}^{n_H-1} v_{(k)}}{n_H-1}\frac{n_H-1}{n_L}$, $C_L = \frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{n_L} - v_{(n-n_L+1)}$ and $C_H = v_{(n-n_L)} + \frac{\sum_{k=1}^{n_H} v_{(k)}}{n_H}\frac{n_H-1}{n_L}$. Then, we can rewrite (53) as

$$\frac{\Pi_{\text{other}}^{\text{after}}(n_L)}{\Pi_{\text{other}}^{\text{before}}(n_L)} = \frac{C'_L + C'_H r}{C_L + C_H r}$$

Two ratios are important for the proof: C'_L/C_L and C'_H/C_H. Using (19), we can rewrite

$$\begin{aligned}
\frac{C'_L}{C_L} &= \frac{\frac{2n-n_L-2}{2} + \alpha\frac{n(n-1)+(2n-n_L-2)(n-n_L-3)}{6} - C_1^{n-n_L-1} - \alpha C_2^{n-n_L-1}}{\frac{2n-n_L-1}{2} + \alpha\frac{n(n-1)+(2n-n_L-1)(n-n_L-2)}{6} - C_1^{n-n_L} - \alpha C_2^{n-n_L}} \\
&= \frac{n_L + \alpha\left(\frac{n(n-1)+(2n-n_L-2)(n-n_L-3)}{3} - (n-n_L-1)(n-n_L-2)\right)}{n_L - 1 + \alpha\left(\frac{n(n-1)+(2n-n_L-1)(n-n_L-2)}{3} - (n-n_L)(n-n_L-1)\right)} \\
&= \frac{n_L + \alpha B_1}{n_L - 1 + \alpha B_2} \qquad (54)
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \frac{n(n-1) + (2n-n_L-2)(n-n_L-3)}{3} - (n-n_L-1)(n-n_L-2) \\
B_2 &= \frac{n(n-1) + (2n-n_L-1)(n-n_L-2)}{3} - (n-n_L)(n-n_L-1)
\end{aligned}$$

The remainder of the proof repeatedly uses the following property: With $X_1, X_2, Y_1, Y_2 > 0$, we have $\frac{X_1}{X_2} < \frac{Y_1}{Y_2}$ if and only if $\frac{X_1+\alpha Y_1}{X_2+\alpha Y_2}$ is increasing in $\alpha > 0$. It is straightforward to verify that

$$\frac{n_L}{n_L-1} > \frac{3n-2n_L-4}{3n-2n_L-2}\frac{n_L}{n_L-1} = \frac{B_1}{B_2}$$

so (54) implies that C'_L/C_L is decreasing in α. Thus,

$$\frac{C'_L}{C_L} \geq \frac{3n-2n_L-4}{3n-2n_L-2}\frac{n_L}{n_L-1} \qquad (55)$$

Using (18), we can rewrite

$$\begin{aligned}
\frac{C'_H}{C_H} &= \frac{\frac{\sum_{k=1}^{n_H-1} v_{(k)}}{n_H-1} + \frac{n_L}{n_H-1} v_{(n-n_L-1)}}{\frac{\sum_{k=1}^{n_H} v_{(k)}}{n_H} + \frac{n_L}{n_H-1} v_{(n-n_L)}} \\
&= \frac{\frac{1}{2}C_1^{n_H-2} + \frac{\alpha}{3}C_2^{n_H-2} + \frac{n-n_H}{n_H-1}(C_1^{n_H-2} + \alpha C_2^{n_H-2})}{\frac{1}{2}C_1^{n_H-1} + \frac{\alpha}{3}C_2^{n_H-1} + \frac{n-n_H}{n_H-1}(C_1^{n_H-1} + \alpha C_2^{n_H-1})} \\
&= \frac{D_1\alpha + E_1}{D_2\alpha + E_2}
\end{aligned} \tag{56}$$

where $E_1 = \frac{1}{2}C_1^{n_H-2} + \frac{n-n_H}{n_H-1}C_1^{n_H-2}$, $E_2 = \frac{1}{2}C_1^{n_H-1} + \frac{n-n_H}{n_H-1}C_1^{n_H-1}$, $D_1 = \frac{1}{3}C_2^{n_H-2} + \frac{n-n_H}{n_H-1}C_2^{n_H-2}$, and $D_2 = \frac{1}{3}C_2^{n_H-1} + \frac{n-n_H}{n_H-1}C_2^{n_H-1}$.

The remainder of the proof consists of four steps. First $C'_L/C_L > E_1/E_2$. Using (55), we have

$$\frac{C'_L}{C_L} \geq \frac{3n-2n_L-4}{3n-2n_L-2}\frac{n_L}{n_L-1} > \frac{\frac{3n-2n_L}{2}-2}{\frac{3n-2n_L}{2}-1} > \frac{n_H-2}{n_H-1} = \frac{E_1}{E_2}$$

where the second inequality is from $\frac{n_L}{n_L-1} > 1$ and the third is from $\frac{3n-2n_L}{2} > n_H$.

Second, $C'_L/C_L > D_1/D_2$. Using the first step, we have

$$\frac{C'_L}{C_L} > \frac{n_H-2}{n_H-1} > \frac{n_H-3}{n_H-1} = \frac{D_1}{D_2}$$

Third, $C'_L/C_L > C'_H/C_H$. Because $\frac{C'_H}{C_H} = \frac{D_1\alpha+E_1}{D_2\alpha+E_2}$, the value of $\frac{C'_H}{C_H}$ is between $\frac{D_1}{D_2}$ and $\frac{E_1}{E_2}$. Therefore, the first two steps imply that $C'_L/C_L > C'_H/C_H$.

Fourth, $\Pi^{\text{after}}_{\text{other}}(n_L)/\Pi^{\text{before}}_{\text{other}}(n_L)$ is decreasing in r. The third step implies $\frac{C'_L+C'_H r}{C_L+C_H r}$ is decreasing in r. ∎

Proof of Proposition 6. Lemmas 2 to 8 imply this proposition. ∎

Lemmas 9 to 11 prove Proposition 7. In particular, Lemma 9 proves the proposition for small α and Lemmas 10 and 11 prove the proposition if α is large enough.

Lemma 9 *If the prize sequence is not very convex, the entry increases the existing players' performance if they are strong enough. That is, ϕ_α is increasing if α is small.*

Proof. It is sufficient to prove the lemma for $\alpha = 0$. First, Lemma 2 implies that $\phi_\alpha(0) = 0$.

Second, according to equation (50), $\Pi^{\text{after}}_{\text{other}}(1)/\Pi^{\text{before}}_{\text{other}}(1) = 1$ implies $r = \frac{1}{n+1}$. Therefore, $\phi_\alpha(\frac{1}{n}) = \frac{1}{n+1}$.

Third, consider case $n_L \geq 2$. The critical value of c_L/c_H has the following expression

$$\phi_\alpha\left(\frac{n_L}{n}\right) = \frac{W'_L\frac{n_L}{n_L+1} - W_L - n_L v_{(n-n_L)} + n_L v_{(n-n_L+1)}}{W_H\frac{n_H-1}{n_H} - W'_H - n_L v_{(n-n_L-1)} + n_L v_{(n-n_L)}} \tag{57}$$

Because $\alpha = 0$, we have $v_{(k)} = k-1$. Therefore, $W'_L = \frac{v_{(n)}+v_{(n-n_L)}}{2}(n_L+1)$, $W_L = \frac{v_{(n)}+v_{(n-n_L+1)}}{2}n_L$,

$W'_H = \frac{v_{(n-n_L-1)}}{2}(n_H-1)$, $W_H = \frac{v_{(n-n_L)}}{2} n_H$. Substituting these expressions into (57), we obtain

$$\phi_\alpha\left(\frac{n_L}{n}\right) = \frac{n_L/2}{(n-1)/2 + n_L/2} \tag{58}$$

which is increasing in n_H.

The combination of $\phi_\alpha(0) = 0$, $\phi_\alpha(\frac{1}{n}) = \frac{1}{n+1}$ and (58) implies that ϕ_α is strictly increasing.

■

Consider the case with large enough α. We want to show that ϕ_α is hump-shaped for $n_L/n = 0, 1/n, ..., (n-1)/n$. Lemma 10 discusses the case with $n_L/n = 2/n, ..., (n-1)/n$, and Lemma 11 discusses the case with $n_L/n = 0, 1/n$.

Lemma 10 *If α is large enough, $\phi_\alpha\left(\frac{n_L}{n}\right)$ is hump-shaped or decreasing for $\frac{2}{n} \leq \frac{n_L}{n} \leq \frac{n-1}{n}$.*

Proof. First, we study $C_H - C'_H$. Using (56), we have

$$\begin{aligned} C_H - C'_H &= \frac{1}{2}C_1^{n_H-1} + \frac{\alpha}{3}C_2^{n_H-1} + \frac{n-n_H}{n_H-1}(C_1^{n_H-1} + \alpha C_2^{n_H-1}) \\ &\quad - \left[\frac{1}{2}C_1^{n_H-2} + \frac{\alpha}{3}C_2^{n_H-2} + \frac{n-n_H}{n_H-1}(C_1^{n_H-2} + \alpha C_2^{n_H-2})\right] \\ &= \frac{1}{2}C_0^{n_H-2} + \frac{\alpha}{3}C_1^{n_H-2} + \frac{n-n_H}{n_H-1}(C_0^{n_H-2} + \alpha C_1^{n_H-2}) \end{aligned}$$

where the second equality is from Pascal's identity.

Second, we study $C'_L - C_L$. Using (19), we have

$$\begin{aligned} C'_L - C_L &= \frac{\sum_{k=n-n_L}^{n} v_{(k)}}{n_L+1} - v_{(n-n_L)} - \frac{\sum_{k=n-n_L+1}^{n} v_{(k)}}{n_L} + v_{(n-n_L+1)} \\ &= \frac{C_2^n - C_2^{n-n_L-1} + \alpha(C_3^n - C_3^{n-n_L-1})}{n_L+1} - (C_1^{n_H-1} + \alpha C_2^{n_H-1}) \\ &\quad - \frac{C_2^n - C_2^{n-n_L} + \alpha(C_3^n - C_3^{n-n_L})}{n_L} + C_1^{n_H} + \alpha C_2^{n_H} \\ &= K_0 + \alpha K_\alpha \end{aligned}$$

where

$$\begin{aligned} K_0 &= \frac{C_2^n - C_2^{n-n_L-1}}{n_L+1} - C_1^{n_H-1} - \frac{C_2^n - C_2^{n-n_L}}{n_L} + C_1^{n_H} \\ K_\alpha &= \frac{C_3^n - C_3^{n-n_L-1}}{n_L+1} - C_2^{n_H-1} - \frac{C_3^n - C_3^{n-n_L}}{n_L} + C_2^{n_H} \\ &= \frac{C_3^n - C_3^{n-n_L-1}}{n_L+1} - \frac{C_3^n - C_3^{n-n_L}}{n_L} + C_1^{n_H-1} \end{aligned}$$

where the last equality is from Pascal's identity.

Third, ϕ_α is hump-shaped. To see this, using (53), we can solve for r such that $\frac{\Pi_{\text{other}}^{\text{after}}(n_L)}{\Pi_{\text{other}}^{\text{before}}(n_L)} = 1$. The solution is

$$r = \frac{C'_L - C_L}{C_H - C'_H}$$

According to its definition, if $\phi_\alpha(n_L/n)$ is interior, it equals to the above solution:

$$\phi_\alpha(n_L/n) = \frac{C'_L - C_L}{C_H - C'_H}$$

The first two steps shows that if α is large enough, $\frac{C'_L - C_L}{C_H - C'_H}$ converges to

$$\begin{aligned}
\frac{K_\alpha}{\frac{1}{3}C_1^{n_H-2} + \frac{n-n_H}{n_H-1}C_1^{n_H-2}} &= \frac{\frac{C_3^n - C_3^{n-n_L-1}}{n_L+1} - \frac{C_3^n - C_3^{n-n_L}}{n_L} + C_1^{n_H-1}}{\frac{1}{3}C_1^{n_H-2} + \frac{n-n_H}{n_H-1}C_1^{n_H-2}} \\
&= \frac{(-n+4n_H-2)(n_H-1)}{2(n_H-2)(3n-2n_H-1)} \qquad (59)
\end{aligned}$$

For $n = 4$ or 5, it is straightforward to verify that the above expression is a U-shaped function for $n_H = 2, ..., n-2$. For $n \geq 6$, the derivative of (59) w.r.t. n_H is proportional to

$$\begin{aligned}
& 10n\left(n_H - \frac{11n-6}{5n}\right)^2 - 10n\left(\frac{11n-6}{5n}\right)^2 + 3n^2 + 25n - 18 \\
\geq\ & -10n\left(\frac{11n-6}{5n}\right)^2 + 3n^2 + 25n - 18 \\
=\ & \frac{3(n-6)(n-1)(5n-4)}{5n} \geq 0
\end{aligned}$$

Thus, for all $n \geq 4$, the expression in (59) is a U-shaped function or increasing function of n_H for $n_H = 2, ..., n-2$, so for a fixed n, it is a hump-shaped or decreasing function of $n_L = n - n_H$ for $n_L = 2, ..., n-2$. Recall that $\phi_\alpha\left(\frac{n_L}{n}\right)$ converges to (59) if $\alpha \to \infty$, so in the limit, we have $\phi_\alpha\left(\frac{n_L}{n}\right)$ is a hump-shaped or decreasing function for $\frac{n_L}{n} = \frac{2}{n}, ..., \frac{n-2}{n}$. Notice that $\phi_\alpha\left(\frac{n-1}{n}\right) = 0$ according to Lemma 3, so $\phi_\alpha\left(\frac{n_L}{n}\right)$ is a hump-shaped or decreasing function for $\frac{2}{n} \leq \frac{n_L}{n} \leq \frac{n-1}{n}$.
■

Lemma 11 *If the prize sequence is sufficiently convex, the entry of an L-cost player decreases the existing players' performance if they are more homogeneous. That is, ϕ_α is hump-shaped if α is large enough.*

Proof. If $n = 3$ and if α is large, $\phi_\alpha(0) = 0$ according to Lemma 2 and $\phi_\alpha(2/n) = 0$ according to Lemma 3, so ϕ_α is hump-shaped.

Suppose $n \geq 4$. Recall that $\phi_\alpha(0) = 0$ according to Lemma 2, so based on Lemma 10, it is sufficient to show $\phi_\alpha(1/n) < \phi_\alpha(2/n)$. We prove it in three steps.

First, $\lim_{\alpha\to+\infty} \hat{x}/v_{(n)} = \frac{1-r}{\frac{n-1}{n-3} - r}$. The definition of $\hat{x}$ implies $\bar{F}_H(\hat{x}) = \hat{F}_H(\hat{x})$. Substituting (31) and (32) into it, we can rewrite this equation as

$$\frac{\frac{-C_1^{n-1} + \sqrt{(C_1^{n-1})^2 + 4\alpha C_2^{n-1}(v_{(n)}(1-r) + r\hat{x})}}{2\alpha C_2^{n-1}}}{\frac{-C_1^{n-2} + \sqrt{(C_1^{n-2})^2 + 4\alpha C_2^{n-2}\hat{x}}}{2\alpha C_2^{n-2}}} = 1$$

If α goes to infinity, the above equation implies

$$\left(\lim_{\alpha\to+\infty}\frac{v_{(n)}}{\hat{x}}(1-r)+r\right)\frac{n-3}{n-1}=1$$

Therefore,

$$\lim_{\alpha\to+\infty}\hat{x}/v_{(n)}=\frac{1-r}{\frac{n-1}{n-3}-r} \tag{60}$$

Second,

$$\lim_{\alpha\to+\infty}\frac{\mathbb{E}[c_H s_H]}{v_{(n)}} = 1-\frac{2}{3r}+\frac{2}{3r}\left(r\frac{1-r}{\frac{n-1}{n-3}-r}-r+1\right)^{\frac{3}{2}}-\frac{2\,(n-3)}{3\,(n-1)}\left(\frac{1-r}{\frac{n-1}{n-3}-r}\frac{n-1}{n-3}\right)^{\frac{3}{2}}$$

To see why, we have

$$\begin{aligned}
\frac{\mathbb{E}[c_H s_H]}{v_{(n)}} &= \int_{\hat{x}}^{v_{(n)}}\frac{t}{v_{(n)}}d\bar{F}_H(t)+\int_0^{\hat{x}}\frac{t}{v_{(n)}}d\hat{F}_H(t)\\
&= \int_{\hat{x}/v_{(n)}}^{1}xd\bar{F}_H(v_{(n)}x)+\int_0^{\hat{x}/v_{(n)}}xd\hat{F}_H(v_{(n)}x)\\
&= 1-\int_{\hat{x}/v_{(n)}}^{1}\bar{F}_H(v_{(n)}x)dx-\int_0^{\hat{x}/v_{(n)}}\hat{F}_H(v_{(n)}x)dx
\end{aligned}$$

where the second equality is from a change of variables and the last is from integration by parts. Substituting (31) and (32) into the above expression, we have

$$\begin{aligned}
&\lim_{\alpha\to+\infty}\frac{\mathbb{E}[c_H s_H]}{v_{(n)}}\\
&= 1-\lim_{\alpha\to+\infty}\int_{\hat{x}/v_{(n)}}^{1}\sqrt{\frac{2v_{(n)}(1-r(1-x))}{\alpha(n-1)(n-2)}}dx-\lim_{\alpha\to+\infty}\int_0^{\hat{x}/v_{(n)}}\sqrt{\frac{2v_{(n)}x}{\alpha(n-2)(n-3)}}dx\\
&= 1-\lim_{\alpha\to+\infty}\int_{\hat{x}/v_{(n)}}^{1}\sqrt{\frac{2\frac{\alpha(n-1)(n-2)}{2}(1-r(1-x))}{\alpha(n-1)(n-2)}}dx-\lim_{\alpha\to+\infty}\int_0^{\hat{x}/v_{(n)}}\sqrt{\frac{2\frac{\alpha(n-1)(n-2)}{2}x}{\alpha(n-2)(n-3)}}dx\\
&= 1-\lim_{\alpha\to+\infty}\int_{\hat{x}/v_{(n)}}^{1}\sqrt{1-r(1-x)}dx-\lim_{\alpha\to+\infty}\int_0^{\hat{x}/v_{(n)}}\sqrt{\frac{n-1}{n-3}}xdx\\
&= 1-\lim_{\alpha\to+\infty}\frac{2}{3r}\left(rx-r+1\right)^{\frac{3}{2}}\Big|_{\hat{x}/v_{(n)}}^{1}-\lim_{\alpha\to+\infty}\frac{2}{3\,(n-1)}(n-3)\left(x\frac{n-1}{n-3}\right)^{\frac{3}{2}}\Big|_0^{\hat{x}/v_{(n)}}\\
&= 1-\lim_{\alpha\to+\infty}\left(\frac{2}{3r}-\frac{2}{3r}\left(r\hat{x}/v_{(n)}-r+1\right)^{\frac{3}{2}}\right)-\lim_{\alpha\to+\infty}\frac{2}{3\,(n-1)}(n-3)\left(\hat{x}/v_{(n)}\frac{n-1}{n-3}\right)^{\frac{3}{2}}
\end{aligned}$$

Substituting (60) into the above equation, we have

$$\begin{aligned}
\lim_{\alpha\to+\infty}\frac{\mathbb{E}[c_H s_H]}{v_{(n)}} &= 1-\frac{2}{3r}+\frac{2}{3r}\left(r\frac{1-r}{\frac{n-1}{n-3}-r}-r+1\right)^{\frac{3}{2}}-\frac{2\,(n-3)}{3\,(n-1)}\left(\frac{1-r}{\frac{n-1}{n-3}-r}\frac{n-1}{n-3}\right)^{\frac{3}{2}}\\
&\equiv B
\end{aligned}$$

Third, $\lim_{\alpha\to+\infty}\phi_\alpha(2/n) \geq \lim_{\alpha\to+\infty}\phi_\alpha(1/n)$. To see this, notice that

$$\begin{aligned}
\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)} &= \lim_{\alpha\to+\infty}\frac{(v_{(n)}-v_{(n-1)})/2+(v_{(n-2)}+\sum_{k=1}^{n-2}v_{(k)})r}{\sum_{k=1}^{n-1}v_{(k)}+(v_{(n)}+(n-2)\mathbb{E}[c_Hs_H])r-(n-1)\mathbb{E}[c_Hs_H]}\\
&= \frac{(C_2^{n-1}-C_2^{n-2})\frac{1}{2}+C_2^{n-3}r+r(C_3^n-(C_2^{n-1}+C_2^{n-2}))}{C_3^n-C_2^{n-1}+(C_2^{n-1}+(n-2)C_2^{n-1}B)r-(n-1)C_2^{n-1}B}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(2)}{\Pi^{\text{before}}_{\text{other}}(2)}\\
=\ &\lim_{\alpha\to+\infty}\frac{(v_{(n)}+v_{(n-1)}+v_{(n-2)})\frac{2}{3}-2v_{(n-2)}+(2v_{(n-3)}+\sum_{k=1}^{n-3}v_{(k)})r}{(v_{(n)}+v_{(n-1)}-2v_{(n-1)})+(2v_{(n-2)}+\frac{n-3}{n-2}\sum_{k=1}^{n-2}v_{(k)})r}\\
=\ &\frac{(C_2^{n-1}+C_2^{n-2}+C_2^{n-3})\frac{2}{3}-2C_2^{n-3}+2rC_2^{n-4}+r(C_3^n-C_2^{n-1}-C_2^{n-2}-C_2^{n-3})}{C_2^{n-1}-C_2^{n-2}+2rC_2^{n-3}+\frac{n-3}{n-2}r(C_3^n-C_2^{n-1}-C_2^{n-2})}
\end{aligned}$$

If $n=4$, $\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(2)}{\Pi^{\text{before}}_{\text{other}}(2)}=4/3$, which means the others' total performances are always encouraged for all values of r. Therefore, $\lim_{\alpha\to+\infty}\phi_\alpha(2/n)=1/2$, which is the upper bound of r. Substituting $r=1/2$ into $\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}$, we have $\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}<1$. Therefore, $\lim_{\alpha\to+\infty}\phi_\alpha(1/n)<1/2$, and $\lim_{\alpha\to+\infty}\phi_\alpha(1/n)<\lim_{\alpha\to+\infty}\phi_\alpha(2/n)$.

If $n>4$, $\lim_{\alpha\to+\infty}\phi_\alpha(2/n)=\frac{3n-10}{n^2-n-12}$. Substituting $r=\frac{3n-10}{n^2-n-12}$ into $\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}$, we can verify that $\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}$ is increasing in n and $\lim_{n\to+\infty}\lim_{\alpha\to+\infty}\frac{\Pi^{\text{after}}_{\text{other}}(1)}{\Pi^{\text{before}}_{\text{other}}(1)}<1$. Therefore, $\lim_{\alpha\to+\infty}\phi_\alpha(1/n)<\lim_{\alpha\to+\infty}\phi_\alpha(2/n)$. ■

Proof of Proposition 7. Lemmas 9 to 11 imply Proposition 7. ■

Example 5 *Suppose $n=10$ and $\alpha=0.2$. Then, $\phi_\alpha(n_L/n;\alpha)$ is neither monotone nor hump-shaped in n_L/n. As in Figure 13, it is increasing for small values of n_L/n, then decreasing for medium values of n_L/n, and eventually increasing for high values of n_L/n.*

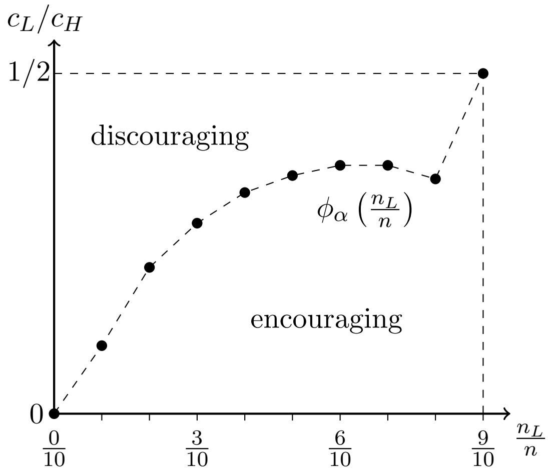

Figure 13: $\alpha = 0.2$

D Omitted Proofs in Section 6

Proof of Proposition 8. Because of Proposition 6, it is sufficient to show that the entrant has higher expected performance than the player that she replaces. We prove in four steps:

Step 1. If $n_L = 0$, no c_L and c_H values satisfy $c_L/c_H \le \phi_\alpha(n_L/n)$. Therefore, the corollary is true.

Step 2. Suppose $n = 2$ and $n_L = 1$ before the entry. After the entry, there are two L-type players, so the entrant's performance is $\frac{V}{2}\frac{1}{c_L}$, which must be higher than an H-cost player's before the entry.

Step 3. Now consider $n \ge 3$ and $n_L = 1$ before the entry. If $n_L = 1$, the performance of an H-cost player is lower than $\frac{V}{n}\frac{1}{c_H}$. If $n_L = 2$, the performance of an L-cost player is $\frac{\frac{v_{(n)}+v_{(n-1)}}{2} - u_L}{c_L}$. Notice that the highest performance of an H-cost player is $\bar{s}_H = v_{(n-2)}/c_H$, then an L-cost player's payoff at $\bar{s}_H$ is $u_L = v_{(n-1)} - c_L\bar{s}_H = v_{(n-1)} - v_{(n-2)}\frac{c_L}{c_H}$. Hence, if $n_L = 2$, an L-cost player's performance is $\frac{\frac{v_{(n)}+v_{(n-1)}}{2} - v_{(n-1)} + v_{(n-2)}r}{c_L} = \left(\frac{v_{(n)}+v_{(n-1)}}{2} - v_{(n-1)}\right)\frac{1}{c_L} + \frac{v_{(n-2)}}{c_H}$. Now, the performance difference between the L-cost player if $n_L = 2$ and the H-cost player if $n_L = 1$ is higher than

$$
\begin{aligned}
& \left(\frac{v_{(n)}+v_{(n-1)}}{2} - v_{(n-1)}\right)\frac{1}{c_L} + \frac{v_{(n-2)}}{c_H} - \frac{V}{n}\frac{1}{c_H} \\
> & \left(\frac{v_{(n)}+v_{(n-1)}}{2} - v_{(n-1)}\right)\frac{2}{c_H} + \left(v_{(n-2)} - \frac{V}{n}\right)\frac{1}{c_H} \\
= & \left(v_{(n)} - v_{(n-1)} + v_{(n-2)} - \frac{V}{n}\right)\frac{1}{c_H}
\end{aligned} \tag{61}
$$

where the inequality is from $c_L < c_H/2$. For $n = 3$, the expression in (61) becomes

$$
\frac{2}{3}\frac{\left(v_{(3)} - v_{(2)}\right) - \left(v_{(2)} - v_{(1)}\right)}{c_H} > 0
$$

For $n \ge 4$, we can rewrite the expression in (61) as

$$
\begin{aligned}
& \frac{(n-1)[(v_{(n)} - v_{(n-1)}) - (v_{(n-1)} - v_{(n-2)})] + (n-3)v_{(n-1)} - \sum_{k=1}^{n-3} v_{(k)}}{nc_H} \\
= & \frac{(n-1)\alpha + \sum_{k=1}^{n-3}(v_{(n-1)} - v_{(k)})}{nc_H}
\end{aligned}
$$

which is also positive.

Step 4. Suppose $n \ge 4$ and $n_L \ge 2$ before the entry. If $n_L \ge 2$, the performance of an H-cost player is $\frac{v_{(n-n_H)}+\ldots+v_{(1)}}{n-n_L}\frac{1}{c_H}$. If there are $n_L + 1$ L-cost players, the performance of an L-cost player is $\frac{v_{(n)}+\ldots+v_{(n-n_L)}}{n_L+1}\frac{1}{c_L} - \frac{v_{(n-n_L)}}{c_L} + \frac{v_{(n-n_L-1)}}{c_H}$. Therefore, the performance difference between the L-cost and H-cost players is

$$\begin{aligned}
& \frac{v_{(n)}+...+v_{(n-n_L)}}{n_L+1}\frac{1}{c_L}-\frac{v_{(n-n_L)}}{c_L}+\frac{v_{(n-n_L-1)}}{c_H}-\frac{v_{(n-n_L)}+...+v_{(1)}}{n-n_L}\frac{1}{c_H} \\
= & \left(\frac{v_{(n)}+...+v_{(n-n_L)}}{n_L+1}-v_{(n-n_L)}\right)\frac{1}{c_L}+\left(v_{(n-n_L-1)}-\frac{v_{(n-n_L)}+...+v_{(1)}}{n-n_L}\right)\frac{1}{c_H} \\
> & \ 0
\end{aligned}$$

where the last inequality comes from the increasing prize sequence. ■

Next, we prove Proposition 9. In particular, Lemma 12 proves the proposition for $n_L \geq 2$, and Lemmas 13 and 14 for $n_L = 1$. Finally, Lemma 15 proves the proposition for $n_L = 0$.

Lemma 12 *If $n_L \geq 2$, we have $\Pi_{all}^{after}(n_L) > \Pi_{all}^{before}(n_L)$.*

Proof. Notice before the entry, there are n_L L-cost players before the entry, and there are n_L+1 after it. Before the entry, the total expected performance is

$$\Pi_{\text{all}}^{\text{before}}(n_L) = \frac{v_{(n)}+...+v_{(n-n_L+1)}-n_L u_L}{c_L}+\frac{V-v_{(n)}-...-v_{(n-n_L+1)}}{c_H}$$

Substituting $u_L = v_{(n-n_L+1)} - c_L\bar{s}_H = v_{(n-n_L+1)} - c_L\frac{v_{(n-n_L)}}{c_H}$ into the equation above, we obtain

$$\begin{aligned}
\Pi_{\text{all}}^{\text{before}}(n_L) &= \frac{v_{(n)}+...+v_{(n-n_L+1)}}{c_L}-\frac{n_L}{c_L}\left(v_{(n-n_L+1)}-c_L\frac{v_{(n-n_L)}}{c_H}\right) \\
&\quad +\frac{V-v_{(n)}-...-v_{(n-n_L+1)}}{c_H}
\end{aligned}$$

After the entry, the total expected performance is

$$\begin{aligned}
\Pi_{\text{all}}^{\text{after}}(n_L) &= \frac{v_{(n)}+...+v_{(n-n_L)}}{c_L}-\frac{n_L+1}{c_L}\left(v_{(n-n_L)}-c_L\frac{v_{(n-n_L-1)}}{c_H}\right) \\
&\quad +\frac{V-v_{(n)}-...-v_{(n-n_L)}}{c_H}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pi_{\text{all}}^{\text{after}}(n_L)-\Pi_{\text{all}}^{\text{before}}(n_L) &= \frac{1}{c_L}n_L(v_{(n-n_L+1)}-v_{(n-n_L)})-\frac{1}{c_H}(n_L+1)(v_{(n-n_L)}-v_{(n-n_L-1)}) \\
&> \frac{2}{c_H}n_L(v_{(n-n_L+1)}-v_{(n-n_L)})-\frac{1}{c_H}(n_L+1)(v_{(n-n_L)}-v_{n-n_L-1}) \\
&= \frac{1}{c_H}\left[2n_L(v_{(n-n_L+1)}-v_{(n-n_L)})-(n_L+1)(v_{(n-n_L)}-v_{(n-n_L-1)})\right] \\
&> 0
\end{aligned}$$

where the first inequality comes from $2c_L < c_H$ and the second from $2n_L > n_L+1$ and $v_{(n-n_L+1)} - v_{(n-n_L)} \geq v_{(n-n_L)} - v_{(n-n_L-1)}$. ■

In the discussion of $n_L = 1$, we first consider the two extreme cases with $\alpha = 0$ and $\alpha \to \infty$, then the cases in between.

Lemma 13 *If $\alpha = 0$ and $n_L = 1$, there is a unique $\hat{r} \in (0, 1/2)$ s.t. $\Pi_{all}^{after}(n_L) > \Pi_{all}^{before}(n_L)$ if and only if $c_L/c_H > \hat{r}$.*

Proof. We prove in three steps. In the first step, we express the difference of the total performance $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1)$ in terms of c_L, n and r. To do so, we first express the difference using $c_H \mathbb{E}[s_H]$ and the prizes, then use Lemma 1 to rewrite the difference.

Before the entry of an L-cost player, there is only one L-cost player in the contest, and the total performance is $\Pi_{\text{all}}^{\text{before}}(1) = \frac{W_L - u_L}{c_L} + \frac{V - W_L}{c_H}$, where W_L is the expected winnings of the L-cost player and u_L is his payoff. Notice that the L-cost and H-cost players's equilibrium strategies have the same upper bound, the L-cost's payoff is $u_L = v_{(n)} - c_L \frac{v_{(n)}}{c_H}$. Therefore, we can rewrite $\Pi_{\text{all}}^{\text{before}}(1)$ as

$$\begin{aligned} \Pi_{\text{all}}^{\text{before}}(1) &= \frac{W_L}{c_L} - \frac{1}{c_L}\left(v_{(n)} - c_L \frac{v_{(n)}}{c_H}\right) + \frac{V - W_L}{c_H} \\ &= \frac{W_L - v_{(n)}}{c_L} + \frac{v_{(n)} + V - W_L}{c_H} \end{aligned} \tag{62}$$

After the entry of an L-cost player, there are two L-cost players. Therefore, the supports of the L-cost and H-cost players do not overlap. Then, the total expected performance is $\Pi_{\text{all}}^{\text{after}}(1) = \frac{v_{(n)} + v_{(n-1)} - 2\hat{u}_L}{c_L} + \frac{V - v_{(n)} - v_{(n-1)}}{c_H}$, where the L-cost's payoff is $\hat{u}_L = v_{(n-1)} - c_L \frac{v_{(n-2)}}{c_H}$. Substituting $\hat{u}_L$ into $\Pi_{\text{all}}^{\text{after}}(1)$, we obtain

$$\Pi_{\text{all}}^{\text{after}}(1) = \frac{v_{(n)} + v_{(n-1)}}{c_L} - \frac{2}{c_L}\left(v_{(n-1)} - c_L \frac{v_{(n-2)}}{c_H}\right) + \frac{V - v_{(n)} - v_{(n-1)}}{c_H} \tag{63}$$

Combining (62) and (63), we have

$$\begin{aligned} & \Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) \\ = & \frac{v_{(n)} - v_{(n-1)} + v_{(n)} - W_L}{c_L} - \frac{v_{(n)} + v_{(n-1)} - 2v_{(n-2)} + v_{(n)} - W_L}{c_H} \\ = & \frac{1}{c_L}\left[v_{(n)} - v_{(n-1)} + v_{(n)} - W_L - r(v_{(n)} + v_{(n-1)} - 2v_{(n-2)} + v_{(n)} - W_L)\right] \\ = & \frac{1}{c_L}\left[v_{(n)} - v_{(n-1)} - r(v_{(n)} + v_{(n-1)} - 2v_{(n-2)}) + (1 - r)(v_{(n)} - W_L)\right] \end{aligned} \tag{64}$$

where $r = c_L/c_H$.

Recall that there is only one L-cost player before the entry, so the total winnings of all $n-1$ H-cost players are $V - W_L = (n-1)W_H = (n-1)c_H \mathbb{E}[s_H]$. Therefore, $W_L = V - (n-1)c_H \mathbb{E}[s_H]$. Substituting the expression of W_L into (64), we obtain

$$\begin{aligned} \Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) &= \frac{v_{(n)} - v_{(n-1)} - r(v_{(n)} + v_{(n-1)} - 2v_{(n-2)})}{c_L} \\ &\quad + (1 - r)\frac{-(V - v_{(n)}) + (n-1)c_H \mathbb{E}[s_H]}{c_L} \end{aligned} \tag{65}$$

If $\alpha = 0$, $v_{(1)} = 0, v_{(2)} = 1, ..., v_{(n)} = n - 1$, so $V = (n-1)n/2$. Substituting these prize values and (34) into (65), we obtain

$$\begin{aligned}
&\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) \\
= \;& \frac{1}{c_L}\left(1-3r+(1-r)\left[-\frac{(n-1)n}{2}+(n-1)+\frac{(n-1)^2}{2}\frac{1-r\left(2-\frac{n-1}{n-2}\right)}{\frac{n-1}{n-2}-r}\right]\right) \qquad (66)
\end{aligned}$$

The above equation provides express $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1)$ in terms of n, c_H and r.

In the second step, we show that $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) < 0$ if $r = 1/2$ and $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) > 0$ if $r \to 0$. To see why, if we substitute $r = 1/2$ into (66), we have $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) = -\frac{n+1}{4nc_L} < 0$. If $r \to 0$, (64) becomes

$$\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) = \frac{1}{c_L}\left[v_{(n)} - v_{(n-1)} + (v_{(n)} - W_L)\right] > 0$$

where the inequality comes from $v_{(n)} - W_L \geq 0$ because the H-type player before entry cannot win $v_{(n)}$ with probability 1. Hence, by the Intermediate Value Theorem, there exists $\hat{r} \in (0, 1/2)$ s.t. $\Pi_{\text{all}}^{\text{after}}(1) = \Pi_{\text{all}}^{\text{before}}(1)$.

In the third step, we show that $\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1)$ is strictly decreasing in r. Taking derivatives of both sides of (66) w.r.t. r, we have

$$\begin{aligned}
\frac{\partial(\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1))}{\partial r} &= -3-\left[-\frac{(n-1)n}{2}+(n-1)+\frac{(n-1)^2}{2}\frac{1-r\left(2-\frac{n-1}{n-2}\right)}{\frac{n-1}{n-2}-r}\right] \\
&\quad +(1-r)\frac{(n-1)^2}{2}\frac{\partial}{\partial r}\left[\frac{1-r\left(2-\frac{n-1}{n-2}\right)}{\frac{n-1}{n-2}-r}\right] \\
&= -3-\left[-\frac{(n-1)n}{2}+(n-1)+\frac{(n-1)^2}{2}\frac{1-r\left(2-\frac{n-1}{n-2}\right)}{\frac{n-1}{n-2}-r}\right] \\
&\quad +(1-r)\frac{(n-1)^2}{2}\frac{\left(1-\frac{n-1}{n-2}\right)^2}{\left(\frac{n-1}{n-2}-r\right)^2}
\end{aligned}$$

where the first two terms are negative and the last is positive. We can rewrite the first and last terms as

$$\begin{aligned}
-3+(1-r)\frac{(n-1)^2}{2}\frac{\left(1-\frac{n-1}{n-2}\right)^2}{\left(\frac{n-1}{n-2}-r\right)^2} &= -3+\frac{1-r}{2}\left(\frac{n-1}{n-2}\right)^2\frac{1}{\left(\frac{n-1}{n-2}-r\right)^2} \\
&= -3+\frac{1-r}{2}\frac{1}{\left(1-\frac{r}{\frac{n-1}{n-2}}\right)^2} \\
&\leq -3+\frac{1-r}{2}\frac{1}{(1-r)^2} \\
&= -3+\frac{1}{2(1-r)} \leq -2 < 0
\end{aligned}$$

where the second inequality comes from $r \in (0, 1/2)$. Therefore, $\frac{\partial}{\partial r}(\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1)) < 0$.

Hence, combining the second and the third step, we prove the lemma. ■

Lemma 14 *If $n_L = 1$ and α is large enough, $\Pi_{all}^{after}(n_L) - \Pi_{all}^{before}(n_L) > 0$.*

Proof. Recall that W_L is the only L-cost player's expected performance before the entry, then equation (64) implies

$$\begin{aligned}
& \Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1) \\
= \; & \frac{1}{c_L}\left[v_{(n)} - v_{(n-1)} - r(v_{(n)} + v_{(n-1)} - 2v_{(n-2)}) + (1-r)(v_{(n)} - W_L)\right] \\
= \; & \frac{V}{c_L}\left(\frac{v_{(n)}}{V}(1-r) - \frac{v_{(n-1)} - r(v_{(n-1)} - 2v_{(n-2)}) + (1-r)(v_{(n)} - W_L)}{V}\right) \\
> \; & \frac{1}{c_L}\left(\frac{v_{(n)}}{V}(1-r) - \frac{v_{(n-1)} - r(v_{(n-1)} - 2v_{(n-2)}) + (1-r)(v_{(n)} - W_L)}{V}\right) \qquad (67)
\end{aligned}$$

where the inequality comes from $V > v_{(2)} = 1$.

If $\alpha \to \infty$, using its expression in (8) we can verify that $\bar{G}'_H(s)$ converges to 0. Notice that $\bar{G}_H$ is the H-cost players' strategies over the common support of both cost types' equilibrium strategies, so the L-cost player wins $v_{(n)}$ with probability 1 in the limit, and therefore $\lim_{\alpha\to\infty} \frac{v_{(n)} - W_L}{V} = 0$. Then, (67) implies

$$\begin{aligned}
& \lim_{\alpha\to\infty} (\Pi_{\text{all}}^{\text{after}}(1) - \Pi_{\text{all}}^{\text{before}}(1)) \\
\geq \; & \frac{1}{c_L}\left(\lim_{\alpha\to\infty}\frac{v_{(n)}}{V}(1-r) - \lim_{\alpha\to\infty}\frac{v_{(n-1)} - r(v_{(n-1)} - 2v_{(n-2)}) + (1-r)(v_{(n)} - W_L)}{V}\right) \\
= \; & \frac{1-r}{c_L}\lim_{\alpha\to\infty}\frac{v_{(n)}}{V} = \frac{1-r}{c_L} > 0
\end{aligned}$$

where the first inequality comes from $\lim_{\alpha\to\infty} v_{(k)}/V = 0$ for $k < n$ and $\lim_{\alpha\to\infty} \frac{v_{(n)} - W_L}{V} = 0$.
■

Lemma 15 *If $n_L = 0$, we have $\Pi_{all}^{after}(n_L) < \Pi_{all}^{before}(n_L)$.*

Proof. Notice that $\Pi_{\text{all}}^{\text{before}}(0) = V/c_H$. In addition, $\Pi_{\text{all}}^{\text{after}}(0) = \frac{W_L - v_{(n)}(1-r)}{c_L} + \frac{V - W_L}{c_H}$. Therefore

$$\begin{aligned} \Pi_{\text{all}}^{\text{after}}(0) - \Pi_{\text{all}}^{\text{before}}(0) &= \frac{W_L - v_{(n)}(1-r)}{c_L} + \frac{V - W_L}{c_H} - \frac{V}{c_H} \\ &= \left(W_L - v_{(n)}\right)\left(\frac{1}{c_L} - \frac{1}{c_H}\right) < 0 \end{aligned}$$

where the inequality comes from $W_L < v_{(n)}$, that is, the L-cost player does not win the highest prize with certainty. ■

Proof of Proposition 9. Lemma 12 proves the proposition for $n_L \geq 2$, Lemmas 13 and 14 for $n_L = 1$, and Lemma 15 for $n_L = 0$.

Next, we first show that ϕ_α is nondecreasing. According to Lemmas 12 and 15, $\phi_\alpha(0) = 0$ and $\phi_\alpha(n_L/n) = 1/2$ for $n_L \geq 2$. Because $\phi_\alpha(1/n) \in [0, 1/2]$, ϕ_α is nondecreasing. ■