# All-Pay Contests with Performance Spillovers*

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**Abstract**

This paper generalizes the results of Siegel (2009, 2010) to accommodate performance spillovers, with which a player's performance in a contest may affect the performance cost of another player. More precisely, we show that, if for any player, the spillovers from other players' performance enter his cost in an additively separable form, then an all-pay contest has a unique Nash equilibrium. Moreover, we construct the equilibrium payoffs and strategies. Both the equilibrium uniqueness and construction are generalized to multiplicatively separable spillovers in a two-player contest.

*JEL classification*: D72, D44, L22

*Keywords*: asymmetric, contest, spillover, unique

## 1 Introduction

Performance spillovers are prevalent in contest situations. For example, higher expenditure from a lobbyist may make it easier for another lobbyist to justify his expenditure; a company's R&D effort may benefit its rivals, and hard working classmates make it easier, or less costly, for an individual student to study hard. Siegel (2009, 2010) studies contests among asymmetric players without spillovers, and Baye, Kovenock and de Vries (2012) study contests between two symmetric players with spillovers. The two setups demonstrate different equilibrium properties. For example, an asymmetric contest without spillovers has a unique Nash equilibrium, while a symmetric contest with spillovers may have one or more Nash equilibria depending on parameter values. To bridge the gap between these studies, this paper investigates contests that allow spillovers among asymmetric players.

Specifically, we introduce two types of spillovers in contests: additive and multiplicative. With additive spillovers, the other players' performance levels enter a player's cost function in an additively separable way. For example, given the other player's performance $s_j$, player

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$i$'s cost of performance $s_i$ is $C_i(s_i, s_j) = s_i - \bar{s}$, which means player $i$'s cost depends on not only his own performance but also the average performance $\bar{s} = (s_i + s_j)/2$. As a result, player $j$'s performance affects $i$'s cost through the average performance.[1] This is an aggregate game with linear structure. Linear models with aggregate performance are widely used in empirical studies of spillovers in innovation (e.g. Audretsch and Feldman (1996)), workplaces (e.g. Mas and Moretti (2009)) and education (e.g. Angrist (2014)). Acemoglu and Jensen (2013) provide a theoretic study on spillovers through the average or aggregate action in more general competitions. These studies focus on competitions that are not based on performance ranking, so those competitions are different from contests.

With multiplicative spillovers, the other players' performance levels affect one player's performance cost in a multiplicatively separable way. An example of such cost functions is $C_i(s_i, s_j) = \bar{s}s_i$, which means the average performance $\bar{s}$ affects the marginal cost of player $i$'s performance $s_i$.[2] Production functions with such a multiplicative form are used in studies of spillovers in R&D (e.g. Griliches (1991)) to capture the aggregate knowledge's effect on an individual firm's marginal productivity. They are also used in studies of more general social interactions, e.g., Glaeser, Scheinkman and Sacerdote (2003).

If we introduce performance spillovers into a contest, the original equilibrium strategies may no longer be an equilibrium.[3] However, we show that all-pay contests with additive or multiplicative spillovers have a unique Nash equilibrium. Moreover, we manage to construct the equilibrium payoffs and strategies. Both equilibrium uniqueness and characterization are useful for applications involving contest design in the presence of spillovers.

## 2 Additive Spillovers

Our model builds on that of Siegel (2010), to which we add the possibility of performance spillovers. Consider a contest in which $n$ risk neutral players compete for $m$ homogeneous monetary prizes, where $0 < m < n$.[4] The prize value is normalized to 1.[5] Denote the set of players as $N = \{1, ..., n\}$. Each player $i$ simultaneously chooses a performance level, or score, $s_i \geq 0$. Let $\mathbf{s} = (s_i)_{i \in N}$ be the scores of all players, and $\mathbf{s}_{-i} = (s_j)_{j \in N \setminus \{i\}}$ be the scores of all players except $i$. Given all players' scores $\mathbf{s}$, player $i$'s payoff is $u_i(\mathbf{s}) = P_i(\mathbf{s}) - C_i(\mathbf{s})$, where $P_i : \mathbb{R}^n_+ \to [0, 1]$ is player $i$'s probability of winning, and $C_i : \mathbb{R}^n_+ \to \mathbb{R}$ is his cost of score. Note

[1] See more in Example 1.

[2] See more in Example 2.

[3] See Examples 1 and 2.

[4] Our results can be extended to heterogeneous prizes. For example, Bulow and Levin (2006) and González-Díaz and Siegel (2013) study contests with arithmetic prize sequences (with constant first order differences), and Xiao (2016) studies contests with quadratic prize sequences (with constant second order differences) or geometric prize sequences (with constant ratios between two consecutive prizes). Equilibrium uniqueness and construction are established in those contests. By the same argument in this paper, we can generalize those results to the case of additively separable spillovers.

[5] Our analysis can be extended to allow players to have asymmetric valuations of the prize.

whether he wins or not, player $i$ incurs the cost.[6] The probability of winning is $P_i(\mathbf{s}) = 1$ if $i$'s score $s_i$ exceeds those of at least $n - m$ other players, $P_i(\mathbf{s}) = 0$ if $s_i$ is lower than those of at least $m$ others, and $P_i(\mathbf{s})$ equals any value in $[0, 1]$ otherwise. For each $i$, $C_i(\mathbf{s})$ is strictly increasing in $s_i$, meaning player $i$'s score $s_i$ is costly for him. Note that $C_i(\mathbf{s})$ depends on all players' scores, so there may be spillovers. If $C_i(\mathbf{s})$ is independent of $\mathbf{s}_{-i}$, there are no spillovers, and our setup reduces to that of Siegel (2010).

We assume that the spillovers from other players' scores enter the cost in an *additively separable* way, i.e., $C_i(\mathbf{s}) = K_i(s_i)+H_i(\mathbf{s}_{-i})$ for each $i$, where $K_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $H_i : \mathbb{R}_+^{n-1} \to \mathbb{R}$ may differ among players, representing asymmetry in costs and spillovers respectively. The contest is of complete information, so these functions are commonly known. Recall that $C_i(\mathbf{s})$ is strictly increasing in $s_i$ and $H_i(\mathbf{s}_{-i})$ is independent of $s_i$, so $K_i(s_i)$ is also strictly increasing in $s_i$. Then, assume that there exists $s_{\max} > 0$ such that $K_i(s_{\max}) > 1$ for all $i$, and define player $i$'s reach as $r_i = K_i^{-1}(1)$, and re-index the players such that $r_1 \geq ... \geq r_n$.[7] We assume $r_i \neq r_{m+1}$ for $i \neq m + 1$. In addition, assume $K_i(0) = 0$ and $K_i$ is continuous and piecewise analytic on $[0, r_{m+1}]$.[8] Moreover, for each $j \neq i$, $H_i(\mathbf{s}_{-i})$ is piecewise continuous in $s_j$ on $[0, r_{m+1}]$.[9] The above contest is referred to as the contest with additive spillover. The following example illustrates the general model in a linear setup.

**Example 1** *Suppose the cost is $C_i(\mathbf{s}) = c_i s_i - h\bar{s}$, where $c_i \in \mathbb{R}_+$ is player $i$'s marginal cost of score, and $\bar{s} = (\sum_{i=1}^{n} s_i)/n$ is the average score. Here the spillover depends on the average score, and $h$ measures the scale of spillover. If $h = 0$, there is no spillover. If $h$ is positive (negative), a higher average score makes player $i$'s score less (more) costly. Assume distinct marginal costs so that $0 < c_1 < ... < c_n$.[10] In this example, $K_i(s_i) = (c_i - h/n)s_i$ and $H_i(\mathbf{s}_{-i}) = -h(\sum_{j\neq i} s_j)/n$. The assumption $\partial C_i(\mathbf{s})/\partial s_i > 0$ requires $h < nc_i$ for all $i$.[11] Both functions depend on the spillover parameter $h$. If $h$ is positive (negative), $K_i(s_i)$ is lower (higher) than player $i$'s scoring cost $c_i s_i$.*

A strategy profile constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players. We only consider Nash equilibria here.

**Equilibrium Characterization** In the absence of spillovers, the method of Siegel (2009) can be used to derive equilibrium payoffs, with which equilibrium strategies can be constructed

[6] Because of the all-pay feature, the cost is sunk, so it remains the same whether a player wins. As a result, the spillovers represented by the cost functions also remain the same whether a player wins or not. In contrast, Baye, Kovenock and de Vries (2012) also consider rank-order spillovers that depend on the rank of a player's score, and demonstrate possibly multiple equilibria in the presence of rank-order spillovers.

[7] The definition of "reach" is first introduced by Siegel (2009).

[8] A function is piecewise analytic on an interval if the interval can be partitioned into a finite number of closed intervals such that the restriction of the function to each interval is analytic.

[9] A function is piecewise continuous on an interval if the function is continuous on all points in the interval except a finite number of points at which the function has finite limits.

[10] This is to ensure the assumption that $r_i \neq r_{m+1}$ for $i \neq m + 1$ is satisfied.

[11] Otherwise, with $h > nc_i$, it is optimal for player $i$ to choose $s_i = +\infty$.

according to the algorithm of Siegel (2010). However, this approach is not applicable here. This is because with spillovers, we can no longer derive equilibrium payoffs as in the case without spillovers.

In contrast to Siegel's method, our method first constructs equilibrium strategies, which we then use to derive equilibrium payoffs. Given the original contest, consider an auxiliary contest with the same prizes but different players, whose cost functions are $K_i(s_i)$ for all $i$. The auxiliary contest has no spillover, but it is different from the original contest without spillovers. For instance, if $h = 0$ in Example 1, there is no spillover, and a player's scoring cost is $c_i s_i$, which is different from $K_i(s_i) = (c_i - h/n)s_i$ in the auxiliary contest.

According to Siegel (2010), the auxiliary contest has a unique equilibrium. In this contest, let $G_i : \mathbb{R}_+ \to [0, 1]$ be the c.d.f. representing player $i$'s equilibrium strategy, and $\mathbf{G} = (G_i)_{i\in N}$ be the equilibrium. If $G_i$ assigns probability 1 to a single score, it represents a pure strategy.

**Lemma 1 (Strategic Equivalence)** *A strategy profile is an equilibrium in the contest with additive spillovers if and only if it is an equilibrium in the auxiliary contest.*

**Proof.** In the auxiliary contest, if the other players use strategies $\mathbf{G}_{-i} = (G_j)_{j\in N\setminus\{i\}}$, player $i$'s expected payoff from choosing $s_i$ is $E[P_i(\mathbf{s}) - K_i(s_i)]$. In the contest with spillovers, if the others players use strategies $\mathbf{G}_{-i}$, player $i$'s expected payoff from choosing $s_i$ becomes $E[P_i(\mathbf{s}) - K_i(s_i)] - E[H_i(\mathbf{s}_{-i})]$, where $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i})d\mathbf{G}_{-i}(\mathbf{s}_{-i})$ is independent of his score.[12] The independence is a result of the additive separability. Thus, $\mathbf{G}$ is also an equilibrium in the contest with spillovers. Similarly, the converse is also true, i.e., any equilibrium in the contest with spillovers is also an equilibrium in the auxiliary contest. ■

The result below shows that the original contest with spillovers also has a unique equilibrium, and it is the same one constructed in the auxiliary contest.

**Proposition 1** *The all-pay contest with additively spillovers has a unique equilibrium, which is the same as the one that the algorithm of Siegel (2010) constructs for the auxiliary contest.*

**Proof.** The strategic equivalence (Lemma 1) implies that $\mathbf{G}$ is also an equilibrium in the contest with spillovers. Moreover, suppose there are multiple equilibria in the contest with spillovers. Then, according to the strategic equivalence, there are also multiple equilibria in the auxiliary contest. This is a contradiction because the auxiliary contest has a unique equilibrium. ■

According to Proposition 1, we can construct the equilibrium in the contest with spillovers as follows: Given any contest with spillovers, find the corresponding auxiliary contest. Then, apply the algorithm of Siegel (2010) to construct the equilibrium in the auxiliary contest, and this constructed equilibrium is also the equilibrium in the contest with spillovers. Below we illustrate the equilibrium construction for Example 1.

[12] According to Siegel (2010), each player $j$'s equilibrium strategy $G_j$ is continuous with a finite support. Moreover, $H_i$ is piecewise continuous so is bounded over the supports of $\mathbf{G}_{-i}$. Hence, $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i})d\mathbf{G}_{-i}(\mathbf{s}_{-i}) < +\infty$.

**Example 1 (continued)** *In the auxiliary contest, player $i$'s cost function is $K_i(s_i) = (c_i - h/n)s_i$. Denote the new marginal cost as $\hat{c}_i \equiv c_i - h/n$. In the equilibrium of this contest characterized by Siegel (2010), player $i = m+2, ..., n$ chooses $s_i = 0$ with probability 1, and player $j = 1, ..., m+1$ mixes over an interval $[s_j^l, s_0^l]$, where $s_m^l = s_{m+1}^l = 0$, $s_0^l = 1/\hat{c}_{m+1}$ and $s_j^l = 1/\hat{c}_{m+1} - \hat{c}_j^{m-j}/(\Pi_{k=j+1}^{m+1}\hat{c}_k)$ for $j = 1, ..., m-1$. Over the interval $[s_j^l, s_{j-1}^l]$, the equilibrium strategy of player $i \in \{j, ..., m+1\}$ is $G_i(s_i) = 1 - (1/\hat{c}_{m+1} - s_i)^{1/(m+1-j)}\beta_{ij}$, where $\beta_{ij} = (\Pi_{k=j}^{m+1}\hat{c}_k^{1/(m+1-j)})/\hat{c}_i$.[13] Proposition 1 implies the above strategy profile is also the unique equilibrium in the original contest with spillovers. Notice that a player's score also affects the average score, so $h$ changes the marginal cost of a player's own score. As a result, $h$ also affects the equilibrium strategies.*

*To illustrate the effect of spillovers through the average score, we compare the equilibrium in the contest with spillovers to the equilibrium in a contest without. Suppose there is one prize and two players, so $m = 1$ and $n = 2$. If $h = 0$, there is no spillover, and the equilibrium strategies are*

$$\begin{aligned} G_1(s; 0) &= c_2 s \\ G_2(s; 0) &= c_1 s + 1 - \frac{c_1}{c_2} \end{aligned}$$

*In contrast, if $h > 0$, there are positive spillovers, and the equilibrium strategies become*

$$\begin{aligned} G_1(s; h) &= (c_2 - h/2)s \\ G_2(s; h) &= (c_1 - h/2)s + 1 - \frac{c_1 - h/2}{c_2 - h/2} \end{aligned}$$

*Note that the spillovers affect the players differently. Player 1's strategy with positive spillovers first order stochastically dominates that without. In contrast, player 2's strategy with spillovers intersects with that without.*

Although the contest with spillovers has the same equilibrium as the auxiliary contest, the associated equilibrium payoff for a player may differ. This is because positive (negative) spillovers from other players' performance bring additional benefits (costs) to a player. In the auxiliary contest, let $\hat{u}_i$ be player $i$'s equilibrium payoff. Then, the result below characterizes the equilibrium payoffs in the contest with spillovers.

**Proposition 2** *In an all-pay contest with additive spillovers, the equilibrium payoff of each player $i$ is $u_i^* = \hat{u}_i - E[H_i(\mathbf{s}_{-i})]$.*

**Proof.** As in the proof of Lemma 1, given the same equilibrium $\mathbf{G}$, the expected payoff of player $i$ in the contest with spillovers is $E[H_i(\mathbf{s}_{-i})]$ lower than that in the auxiliary contest. Hence, $u_i^* = \hat{u}_i - E[H_i(\mathbf{s}_{-i})]$. ■

[13] The expressions of $G_i(s_i)$ and $s_j^l$ are obtained by substituting $V_i = 1, \gamma_i = \hat{c}_i, \alpha = 1, c(y) = y$ and $a_i = \hat{c}_i$ into (11) and (12) of Siegel (2010).

**Example 1 (continued)** *Here we derive the equilibrium payoffs for Example 1. In the auxiliary contest, according to Siegel (2009), the equilibrium payoffs are* $\hat{u}_i = 1 - \hat{c}_i/\hat{c}_{m+1} = 1 - (nc_i - h)/(nc_{m+1} - h)$ *for players* $i = 1, ..., m$, *and* $\hat{u}_i = 0$ *for players* $i = m+1, ..., n$. *Recall that* $H_i(\mathbf{s}_{-i}) = -h(\sum_{j\neq i} s_j)/n$, *so Proposition 2 implies that the equilibrium payoffs in the contest with spillovers are* $u_i^* = \hat{u}_i + h(\sum_{j\neq i} E[s_j])/n$, *where* $E[s_j]$ *is player* $j$*' expected scores in the equilibrium constructed above.*

*Next, we compare the equilibrium payoffs with spillovers to those without. Suppose* $m = 1$ *and* $n = 2$. *If* $h = 0$, *there is no spillover, and the equilibrium payoffs are*

$$\begin{aligned} u_1(0) &= 1 - c_1/c_2 \\ u_2(0) &= 0 \end{aligned}$$

*If* $h > 0$, *there is positive spillover, and the equilibrium payoffs become*

$$\begin{aligned} u_1(h) &= 1 - \frac{c_1 - h/2}{c_2 - h/2} + \frac{h(c_1 + c_2 - h)}{4(c_2 - h/2)^2} \\ u_2(h) &= \frac{h(c_1 + c_2 - h)}{4(c_2 - h/2)^2} \end{aligned}$$

*Notice that the positive spillovers increase both players' equilibrium payoffs, but it increases player 1's payoff more than player 2's. If* $h < 0$, *there is negative spillover, and player 2's equilibrium payoff* $u_2(h)$ *becomes negative.*

## 3 Multiplicative Spillovers

Economists study two-player contests in many contexts including internal labor market tournaments (e.g. Lazear and Rosen (1981)), military conflicts (e.g. Fearon (1995)), and political campaigns (e.g. Che and Gale (1998)).[14] In this section, we consider a contest with one prize of value 1 and two players 1 and 2, where we use $i$ to represent one player and $j$ the other. The model is the same as in Section 2 except that the spillover from the other player's performance enters a player's performance cost in a *multiplicatively separable* way, i.e., $C_i(\mathbf{s}) = K_i(s_i) + L_i(s_i)Q_i(s_j)$ for each $i$, where $K_i, L_i : \mathbb{R}_+ \to \mathbb{R}_+$ represent asymmetry in costs and $Q_i : \mathbb{R}_+ \to \mathbb{R}_+$ represents asymmetry in spillovers.[15]

We assume that there is $s_{\max} > 0$ such that $K_i(s_{\max}) > 1$, so it is never optimal for a player to choose a score above $s_{\max}$. In addition, $K_i$, $L_i$ and $Q_i$ are continuous, and $K_i$ and $L_i$ are piecewise analytic and strictly increasing over $[0, s_{\max}]$. We also assume $K_i(0) = L_i(0) = 0$, so it is costless to choose score 0. Notice that $K_i$ and $L_i$ are strictly increasing and $Q_i(s_j)$ is nonnegative, so $C_i(\mathbf{s}) = K_i(s_i) + L_i(s_i)Q_i(s_j)$ is strictly increasing in $s_i$. There are positive spillovers if $Q_i(s_j)$ is decreasing in $s_j$, negative spillovers if it is increasing in $s_j$, and no spillovers

[14] See Chapter 1.2 of Konrad (2009) for more applications.

[15] The term $K_i(s_i)$ is needed to accommodate the cost function in Example 2. However, the analysis below also applies to $C_i(\mathbf{s}) = L_i(s_i)Q_i(s_j)$, where $Q_i(s_j) \geq \underline{q}_i > 0$.

if it is constant. The above game is referred to as the two-player contest with multiplicative spillovers.

**Example 2** *Suppose each player $i$ has a cost function $C_i(s_1, s_2) = c_i s_i(\alpha + \beta \bar{s})$, where $\bar{s}$ is the average score and $c_i, \beta > 0, \alpha \geq 0$. Then, the average or the aggregate score affects a player's marginal scoring cost. In this example, $C_i(s_1, s_2) = K_i(s_i) + L_i(s_i)Q_i(s_j)$, where $K_i(s_i) = \alpha c_i s_i + \beta c_i s_i^2/2$, $L_i(s_i) = \beta c_i s_i/2$ and $Q_i(s_j) = s_j$.*

Due to the multiplicative part $L_i(s_i)Q_i(s_j)$ in the cost function, the method for additive spillovers no longer applies. To see why, recall that a contest with additive spillovers is strategic equivalent to an auxiliary contest without spillovers. As a result of the strategic equivalence, we can use the results in the contests without spillovers, such as equilibrium uniqueness and construction, to analyze the contests with additive spillovers. In contrast, the strategic equivalence no longer holds for multiplicative spillovers because it is no longer straightfoward to find the auxiliary contest. Indeed, we need to solve a fixed point of a self-mapping to find the auxiliary contest, which makes the analysis in this section much more involved than that in the previous section.[16]

Next, we introduce an auxiliary contest and use it to define a self-mapping, which is used to characterize the equilibrium. Denote $\bar{q}_i = \max_{s_j \in [0, s_{\max}]} Q_i(s_j)$. Given any $(q_1, q_2) \in [0, \bar{q}_1] \times [0, \bar{q}_2]$, consider an auxiliary contest with the same prize and two players with cost functions $K_i(s_i) + L_i(s_i)q_i$ for $i = 1, 2$. Because $q_i$ is constant, there is no spillover in the auxiliary contest. As a result, we can use the method of Siegel (2010) to derive the unique equilibrium. Specifically, player $i$'s reach $r_i(q_i)$ solves $K_i(r_i(q_i)) + L_i(r_i(q_i))q_i = 1$ and the threshold is $T(q_1, q_2) = \min(r_1(q_1), r_2(q_2))$. Then, the equilibrium payoff is $u_i(q_1, q_2) = 1 - K_i(T) - L_i(T)q_i$ for player $i = 1, 2$. The equilibrium strategy $G_i(\cdot; q_1, q_2)$ satisfies

$$G_i(s_j; q_1, q_2) - K_j(s_j) - L_j(s_j)q_j = u_j(q_1, q_2) \tag{1}$$

which means given $G_i(\cdot; q_1, q_2)$, player $j$ receives his equilibrium payoff by choosing $s_j$. Therefore, $G_i(s; q_1, q_2) = u_j(q_1, q_2) + K_j(s_j) + L_j(s_j)q_j$. Then, we can define a mapping $\Phi : [0, \bar{q}_1] \times [0, \bar{q}_2] \to [0, \bar{q}_1] \times [0, \bar{q}_2]$ such that

$$\Phi(q_1, q_2) = (E[Q_1(s_2)|G_2(\cdot; q_1, q_2)], E[Q_2(s_1)|G_1(\cdot; q_1, q_2)])$$

where

$$E[Q_i(s_j)|G_j(\cdot; q_1, q_2)] = \int_0^T Q_i(s) dG_j(s; q_1, q_2) \tag{2}$$

is the expectation of $Q_i(s_j)$ given player $j$'s mixed strategy $G_j(\cdot; q_1, q_2)$.

[16] More precisely, we define a family of auxiliary contests with parameters $(q_1, q_2)$. After finding the unique fixed point $(q_1^*, q_2^*)$, we can pin down an auxiliary contest using the parameter values $(q_1^*, q_2^*)$.

For $i = 1, 2$, let $\hat{r}_i$ be the unique solution of

$$\frac{1 - K_i(\hat{r}_i)}{L_i(\hat{r}_i)} \left(1 - \int_0^{\hat{r}_i} Q_i(s) dL_i(s)\right) = \int_0^{\hat{r}_i} Q_i(s) dK_i(s) \tag{3}$$

in the interval $[0, s_{\max}]$.[17] Without loss of generality, rename the players so that $\hat{r}_1 \geq \hat{r}_2$. The following result establishes a link between equilibria in the the original contest and fixed points of mapping $\Phi$.

**Lemma 2** *If $\{G_1, G_2\}$ is an equilibrium in the two-player contest with multiplicative spillovers, then $(E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_2])$ is a fixed point of $\Phi$. If $(q_1, q_2)$ is a fixed point of $\Phi$, then $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ is an equilibrium of the contest.*

**Proof.** Suppose $\{G_1, G_2\}$ is an equilibrium of the original contest with multiplicative spillovers. Given the other player's strategy $G_j$, player $i$'s payoff from choosing $s_i$ in the original contest is $G_j(s_i) - K_i(s_i) - L_i(s_i)E[Q_i(s_j)|G_j]$, which is the same as $G_j(s_i) - K_i(s_i) - L_i(s_i)q_i$, his payoff from choosing $s_i$ in auxiliary contest with $q_1 = E[Q_1(s_2)|G_2]$ and $q_2 = E[Q_2(s_1)|G_1]$. Hence, $\{G_1, G_2\}$ is also an equilibrium in the auxiliary contest. Then, the definition of $\Phi$ implies $\Phi(q_1, q_2) = (E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_1])$, so $(E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_1])$ is a fixed point of $\Phi$.

Suppose $(q_1, q_2)$ is a fixed point of $\Phi$. In the auxiliary contest, $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ is an equilibrium, which means, given the other player's strategy $G_j(\cdot; q_1, q_2)$, player $i$ does not deviate from $G_i(\cdot; q_1, q_2)$. Then, given the other's strategy $G_j(\cdot; q_1, q_2)$, player $i$'s payoff from choosing $s_i$ in the original contest is $G_j(s_i, q_1, q_2) - K_i(s_i) - L_i(s_i)E[Q_i(s_j)|G_j(\cdot; q_1, q_2)]$, which equals to $G_j(s_i, q_1, q_2) - K_i(s_i) - L_i(s_i)q_i$, his payoff from choosing $s_i$ in the auxiliary contest, because $E[Q_i(s_j)|G_j(\cdot; q_1, q_2)] = q_i$ due to the definition of fixed points. Therefore, player $i$ does not deviate from $G_i(\cdot; q_1, q_2)$ either, which means $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ is an equilibrium in the original contest. ■

As a result, if we find a unique fixed point of $\Phi$, we also find a unique equilibrium in the original contest. The following result characterizes the unique equilibrium.

**Proposition 3** *The two-player contest with multiplicative spillovers has a unique equilibrium. In the equilibrium, the payoff is $u_1^* = 1 - K_1(\hat{r}_2) - L_1(\hat{r}_2)q_1^*$ for player 1 and $u_2^* = 0$ for player 2, player $i$'s strategy is $G_i(s_i) = u_j^* + K_j(s_i) + L_j(s_i)q_j^*$ for $s_i \in [0, \hat{r}_2]$, where*

$$q_1^* = \frac{Q_1(0)(1 - K_1(\hat{r}_2)) + \int_0^{\hat{r}_2} Q_1(s)dK_1(s)}{1 + L_1(\hat{r}_2)Q_1(0) - \int_0^{\hat{r}_2} Q_1(s)dL_1(s)} \tag{4}$$

$$q_2^* = \frac{\int_0^{\hat{r}_2} Q_2(s)dK_2(s)}{1 - \int_0^{\hat{r}_2} Q_2(s)dL_2(s)} \tag{5}$$

[17] To see why there is a unique solution, notice that if $\hat{r}_i \to 0$, the left hand side (LHS) of (3) goes to $+\infty$, which is larger than the right hand side (RHS). In contrast, if $\hat{r}_i = s_{\max}$, the LHS is negative therefore smaller than the RHS. Hence, (3) has at least one solution. Notice that the LHS of the above equation is strictly decreasing in $\hat{r}_i$ while the RHS is weakly increasing, so (3) has a unique solution.

Although $L_i$ and $K_i$ may not be differentiable, the integrals are well-defined because $L_i$ and $K_i$ are piecewise analytic.

**Proof.** We first verify the above equilibrium. In the auxiliary contest with $q_1^*$ and $q_2^*$, the equilibrium strategy $G_i$ is as described in the proposition. Then,

$$\begin{aligned}
E[Q_1(s_2)|G_2] &= \int_0^{\hat{r}_2} Q_1(s_2)dG_2(s_2) \\
&= Q_1(0)G_2(0) + \int_0^{\hat{r}_2} Q_1(s_2)d(K_1(s_2) + q_1^*L_1(s_2)) \\
&= Q_1(0)(1 - K_1(\hat{r}_2) - L_1(\hat{r}_2)q_1^*) + \int_0^{\hat{r}_2} Q_1(s)d(K_1(s) + q_1^*L_1(s)) \qquad (6) \\
&= q_1^*
\end{aligned}$$

where the second equality is from the possible atom of $G_2$ at 0, the third from $G_2(0) = u_1^* = 1 - K_1(\hat{r}_2) - L_1(\hat{r}_2)q_1^*$, and the last from (4).[18] In addition,

$$E[Q_2(s_1)|G_1] = \int_0^{\hat{r}_2} Q_2(s)d(K_2(s) + q_2^*L_2(s)) = q_2^*$$

where the second equality is from (5). Hence, $(q_1^*, q_2^*)$ is a fixed point of $\Phi$, so, according to Lemma 2, the above strategies constitute an equilibrium in the original contest.

To prove the equilibrium uniqueness in the original contest, it is sufficient to show that $\Phi$ has a unique fixed point. Suppose $(q_1, q_2)$ is a fixed point of $\Phi$. First, we show that $r_1(q_1) \geq r_2(q_2)$ in the auxiliary contest with $q_1$ and $q_2$. To see why, suppose otherwise that $r_1(q_1) < r_2(q_2)$. Then, in the auxiliary contest, the threshold is $T = \min\{r_1(q_1), r_2(q_2)\} = r_1(q_1)$ and the equilibrium payoffs are $u_1(q_1, q_2) = 0$ and $u_2(q_1, q_2) = 1 - K_2(r_1(q_1)) - L_2(r_1(q_1))q_2$. Moreover, (1) implies the equilibrium strategies are

$$\begin{aligned}
G_1(s; q_1, q_2) &= K_2(s) + L_2(s)q_2 + u_2(q_1, q_2) \\
G_2(s; q_1, q_2) &= K_1(s) + L_1(s)q_1
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[Q_1(s_2)|G_2(\cdot; q_1, q_2)] &= \int_0^{r_1(q_1)} Q_1(s)d(K_1(s) + q_1L_1(s)) \\
E[Q_2(s_1)|G_1(\cdot; q_1, q_2)] &= Q_2(0)u_2(q_1, q_2) + \int_0^{r_1(q_1)} Q_2(s)d(K_2(s) + q_2L_2(s_2))
\end{aligned}$$

where the second equation is from the same calculation to obtain (6). Hence, $(q_1, q_2) = \Phi(q_1, q_2)$ is equivalent to

$$\begin{aligned}
q_1 &= \int_0^{r_1(q_1)} Q_1(s)d(K_1(s) + q_1L_1(s)) \qquad (7) \\
q_2 &= Q_2(0)u_2(q_1, q_2) + \int_0^{r_1(q_1)} Q_2(s)d(K_2(s) + q_2L_2(s)) \qquad (8)
\end{aligned}$$

[18] The strategy $G_2$ has an atom at 0, i.e., $G_2(0) > 0$, if and only if $u_1^* > 0$.

where $u_2(q_1, q_2) = 1 - K_2(r_1(q_1)) - L_2(r_1(q_1))q_2$. Recall that the definition of $r_i(q_i)$ requires $K_i(r_i(q_i)) + L_i(r_i(q_i))q_i = 1$, so we can express $q_i$ as a function of $r_i(q_i)$. Substituting this expression of $q_i$ into (7) and (8) and rearranging terms, we obtain an equation system about $r_1(q_1)$ and $r_2(q_2)$:

$$\frac{1-K_1(r_1)}{L_1(r_1)}\left(1-\int_0^{r_1} Q_1(s)dL_1(s)\right) = \int_0^{r_1} Q_1(s)dK_1(s) \tag{9}$$

$$\begin{aligned}\frac{1-K_2(r_2)}{L_2(r_2)}\left(1-\int_0^{r_1} Q_2(s)dL_2(s)\right) &= Q_2(0)\left(1-K_2(r_1)-L_2(r_1)\frac{1-K_2(r_2)}{L_2(r_2)}\right)\\ &\quad+\int_0^{r_1} Q_2(s)dK_2(s)\end{aligned} \tag{10}$$

where we omit the arguments of $r_1(q_1)$ and $r_2(q_2)$. Notice that (9) is the same as (3), so $r_1(q_1) = \hat{r}_1$. Recall that $\hat{r}_1 \geq \hat{r}_2$ and $r_1(q_1) < r_2(q_2)$, so $r_2(q_2) > r_1(q_1) \geq \hat{r}_2$. Therefore,

$$\text{Right hand side of (10)} \quad \geq \quad \int_0^{r_1} Q_2(s)dK_2(s) \geq \int_0^{\hat{r}_2} Q_2(s)dK_2(s)$$

where the first inequality is from $1 - K_2(r_1) - L_2(r_1)(1 - K_2(r_2))/L_2(r_2) > 0$ ensured by $r_2(q_2) > r_1(q_1)$, and the second inequality is due to $r_1(q_1) \geq \hat{r}_2$. In addition, $r_2(q_2) > r_1(q_1) \geq \hat{r}_2$ implies $(1 - K_2(\hat{r}_2))/L_2(\hat{r}_2) > (1 - K_2(r_2))/L_2(r_2)$. Thus,

$$\text{Left hand side of (10)} \quad < \quad \frac{1-K_2(\hat{r}_2)}{L_2(\hat{r}_2)}\left(1-\int_0^{\hat{r}_2} Q_2(s)dL_2(s)\right) = \int_0^{\hat{r}_2} Q_2(s)dK_2(s)$$

where the equality is from the definition of $\hat{r}_2$ in (3). Therefore, (10) is violated, which is a contradiction.

As a result, we must have $r_1(q_1) \geq r_2(q_2)$ if $(q_1, q_2)$ is a fixed point. Then, in the auxiliary contest, the threshold is $T = r_2(q_2)$. Repeating the above analysis, we obtain an analogue of (9),

$$\frac{1-K_2(r_2(q_2))}{L_2(r_2(q_2))}\left(1-\int_0^{r_2(q_2)} Q_2(s)dL_2(s)\right) = \int_0^{r_2(q_2)} Q_2(s)dK_2(s)$$

which is the same as (3), so $r_2(q_2) = \hat{r}_2$. Then, repeating the analysis that derives (7) and (8), we obtain their analogues:

$$\begin{aligned} q_2 &= \int_0^{\hat{r}_2} Q_2(s)d(K_2(s) + q_2L_2(s)) \\ q_1 &= Q_1(0)[1 - K_1(\hat{r}_2) - L_1(\hat{r}_2)q_1] + \int_0^{\hat{r}_2} Q_1(s)d(K_1(s) + q_1L_1(s)) \end{aligned}$$

whose unique solution is $(q_1^*, q_2^*)$ described in (4) and (5). Therefore, there cannot be other fixed points of $\Phi$. ■

**Example 2 (continued)** *Next, we derive the equilibrium strategies for* $c_1 = (\sqrt{105} - 9)/4$,

*$c_2 = \sqrt{105} - 9$, $\alpha = 0$ and $\beta = 1$. With these parameter values, $C_i(s_1, s_2) = c_i s_i(\alpha + \beta\bar{s}) = c_i s_i \bar{s} = c_i s_i^2/2 + c_i s_i s_j/2$, so $K_i(s_i) = c_i s_i^2/2$, $L_i(s_i) = c_i s_i/2$ and $Q_i(s_j) = s_j$. Substituting the function forms into (3), we obtain*

$$\frac{2 - c_i\hat{r}_i^2}{c_i\hat{r}_i}\left(1 - \frac{c_i\hat{r}_i^2}{4}\right) = \frac{c_i\hat{r}_i^3}{3}$$

*whose solutions are $\hat{r}_1 = 2$ and $\hat{r}_2 = 1$. Then, we can rewrite (4) and (5) as*

$$q_i^* = \frac{\int_0^1 (c_i s^2) ds}{1 - \int_0^1 (c_i s/2) ds} = \frac{c_i/3}{1 - c_i/4}$$

*for $i = 1, 2$. Therefore, according to Proposition 3, the equilibrium payoffs are $u_1^* = 1 - K_1(1) - L_1(1)q_1^* = 1 - \frac{c_1(12+c_1)}{6(4-c_1)} \approx 0.83$ and $u_2^* = 0$, and the equilibrium strategies are*

$$\begin{aligned} G_1^*(s) &= \frac{c_2}{2}s^2 + \frac{c_2}{2}\frac{c_2/3}{1 - c_2/4}s \\ G_2^*(s) &= \frac{c_1}{2}s^2 + \frac{c_1}{2}\frac{c_1/3}{1 - c_1/4}s + u_1^* \end{aligned}$$

**Three or More Players** With multiplicative spillovers, if there are $m \geq 1$ identical prizes and $n \geq 3$ players, we can no longer solve the equilibrium payoffs and strategies explicitly using the above method. Given $q_1, ..., q_n$, we can still introduce an auxiliary contest similarly. However, two difficulties arise: First, unlike the two-player case, some players may choose zero score with certainty, and it is difficult to determine who they are.[19] Second, if we can determine the actively competing players, the equilibrium strategies in the auxiliary contest may be very complicated. For example, Siegel (2010) shows gaps in the support of the mixed strategies. This makes it difficult to derive the explicit expression of $\Phi$ as in (7) and (8), so it is difficult to solve the fixed points of $\Phi$.

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[19] For some values of $q_1, ..., q_n$, there may be multiple players's reaches equal to the threshold, which violates Assumption B3 of Siegel (2010). Therefore, we cannot pin down the equilibrium payoffs and strategies in the auxiliary contest.

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