Online Appendix of "Bargaining Orders in a Multi-Person Bargaining Game"

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Abstract

This appendix contains the omitted proof of Proposition 1 of Xiao (2017), "Bargaining Orders in a Multi-Person Bargaining Game".

1 The Proof of Proposition 1 for $N = 2$

To distinguish from the lemmas in the main paper, we use claims in this appendix. The proof of Proposition 1 relies on two types of inductions. The first induction is on N, the number of sellers. More precisely, Claims 1 proves the case of $N = 1$, and Claims 2-9 prove the case of $N = 2$. Then, given the proposition for N, we prove for $N + 1$.

For a given a number of sellers, we use a second induction on the horizon T. For example, in the two-seller game with an even horizon, Claim 2 proves Proposition 1 for $T = 2$, and Claims 3-5 prove the proposition for $T = 2t + 2$ given it holds for $T = 2t$. In the two-seller game with an odd horizon, Claim 7 proves for $T = 3$, then Claims 8-9 prove $T = 2t + 3$ from $T = 2t + 1$. We discuss even horizons and odd horizons separately because equilibria evolve differently as T increases.

Claim 1 *In the one-seller game with seller* 1*, the mall is built if and only if* $T \geq 2$*. If the mall is built,*

i) the buyer's equilibrium payoff is $\pi^1_{B,T} = \alpha^1_{1,T}(1 - v_1)$ *with* $\alpha^1_{1,T} \in (0, 1)$

ii) $\alpha^1_{1,2} = 1 - \delta$, $\alpha^1_{1,3} = 1 - \delta + \delta^2$ *and* $\alpha^1_{1,T+2} = 1 - \delta + \delta^2\alpha^1_{1,T}$

iii) $\pi^1_{B,T+2} = (1 - v_1) - \delta(1 - v_1) + \delta^2\pi^1_{B,T}$

iv) $\pi^1_{B,T} < \pi^1_{B,T+2}$ *if* T *is even, and* $\pi^1_{B,T} > \pi^1_{B,T+2}$ *if* T *is odd*

Proof. The game is an alternating-offer bargaining game between the buyer and the seller. If $T = 1$, the buyer offers $p^1_{1,1} = v_1$ to seller 1, and seller 1 accepts. As a result, if seller 1 chooses to participate, his surplus is 0, so he chooses not to participate. Therefore, the mall is not built if $T = 1$.

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Consider the game with $T = 2t + 2 > 1$ periods, $G(1, 2t + 2)$. The buyer offers in the first period, and the alternating offering structure implies that seller 1 offers in the last period. Therefore, we have

$$\begin{aligned} p^1_{1,2t+2} &= H_{1,1} + \delta q^1_{1,2t+1} && (1) \\ 1 - q^1_{1,2t+1} &= \delta(1 - p^1_{1,2t}) && (2) \end{aligned}$$

where (1) means that the buyer suggests a price of $p^1_{1,2t+2}$ such that the seller is indifferent between accepting and rejecting, and (2) means that the seller suggests a price of $q^1_{1,2t+1}$ such that the buyer is indifferent between accepting and rejecting.

Recall that $H_{1,1} = v_1(1-\delta)$, so we can express $p^1_{1,2t+2}$ in terms of $p^1_{1,2t}$ by solving for $q^1_{1,2t+1}$ from (2) and substituting it into (1). In particular, we obtain

$$p^1_{1,2t+2} - v_1 = (1 - v_1)(\delta - \delta^2) + \delta^2(p^1_{1,2t} - v_1) \quad (3)$$

If the one-seller game has only two periods, the equilibrium price is $p^1_{1,2} = v_1 + \delta(1 - v_1)$ by backward induction. We can rearrange this to get $p^1_{1,2} - v_1 = \delta(1 - v_1)$, then solve equation (3) recursively to obtain $p^1_{1,2t+2} - v_1 = (1 - v_1)(\delta - \delta^2 + \delta^3 - ... - \delta^{2t} + \delta^{2t+1})$. Hence,

$$\begin{aligned} p^1_{1,2t+2} &= v_1 + \frac{\delta + \delta^{2t+2}}{1+\delta}(1 - v_1) && (4) \\ &= v_1 + (1 - \alpha^1_{1,2t+2})(1 - v_1) \end{aligned}$$

where $\alpha^1_{1,2t+2} = 1 - (\delta + \delta^{2t+2})/(1 + \delta)$. Therefore, the buyer's payoff is $\pi^1_{B,2t+2} = 1 - p^1_{1,2t+2} = \alpha^1_{1,2t+2}(1 - v_1)$. In addition, $\lim_{t\to\infty} \alpha^1_{1,2t} = \delta/(1-\delta)$, $\alpha^1_{1,2t+2} = 1 - \delta + \delta^2\alpha^1_{1,2t}$, which implies iii), and $\alpha^1_{1,2t} < \alpha^1_{1,2t+2}$, which implies iv).

Similarly, the buyer offers in the final period of $G(1, 2t + 1)$, therefore we have

$$\begin{aligned} p^1_{1,2t+1} &= H_{1,1} + \delta q^1_{1,2t}, \\ 1 - q^1_{1,2t} &= \delta(1 - p^1_{1,2t-1}), \end{aligned}$$

and

$$\begin{aligned} p^1_{1,2t+1} &= v_1 + \frac{\delta - \delta^{2t+1}}{1+\delta}(1 - v_1) && (5) \\ &= v_1 + (1 - \alpha^1_{1,2t+1})(1 - v_1) \end{aligned}$$

where $\alpha^1_{1,2t+1} = 1 - (\delta - \delta^{2t+1})/(1 + \delta)$. Therefore, the buyer's payoff is $\pi^1_{B,2t+1} = 1 - p^1_{1,2t+1} = \alpha^1_{1,2t+1}(1 - v_1)$. In addition, $\lim_{t\to\infty} \alpha^1_{1,2t-1} = \delta/(1-\delta)$, $\alpha^1_{B,2t+1} = 1 - \delta + \delta^2\alpha^1_{1,2t-1}$, which implies iii), and $\alpha^1_{1,2t-1} > \alpha^1_{1,2t+1}$, which implies iv). ■

According to Claim 1, the buyer's payoff in a one-seller game is decreasing in odd T but increasing in even T. Moreover, $\pi^1_{B,3} > ... > \lim_{t\to\infty} \pi^1_{B,2t+1} = \lim_{t\to\infty} \pi^1_{B,2t} > ... > \pi^1_{B,2}$, and

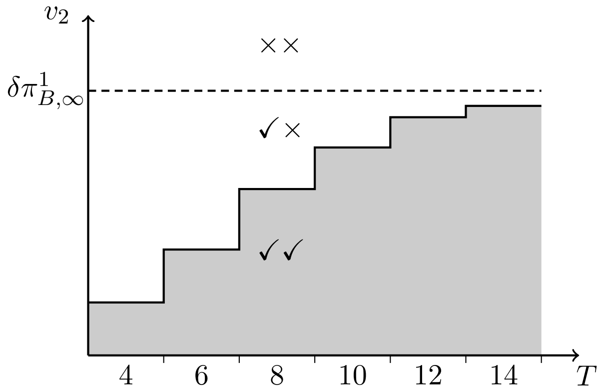

Figure 1: Even Horizons

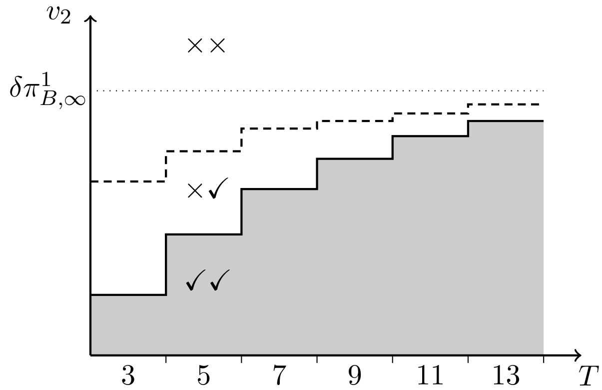

Figure 2: Odd Horizons

$p^1_{1,2} > p^1_{1,4} > ... > \lim_{t\to\infty} p^1_{1,2t} = \lim_{t\to\infty} p^1_{1,2t-1} > ... > p^1_{B,3}$.

Recall that before the bargaining game starts, every player chooses not to participate if his/her surplus in every equilibrium is zero. Next, suppose every player chooses to participate the bargaining game and consider the resulting subgame, which we refer to as the "two-seller game given participation". In the two-seller game given participation, consider four cases:

"××": no agreement in period 1 of $G(B, 2, T)$ and $G(2, B, T-1)$

"✓×": agreement in period 1 of $G(B, 2, T)$ but no agreement in period 1 of $G(2, B, T-1)$

"×✓": no agreement in period 1 of $G(B, 2, T)$ but agreement in period 1 of $G(2, B, T-1)$

"✓✓": agreement in period 1 of $G(B, 2, T)$ and $G(2, B, T-1)$

According to Claim 3, case "×✓" does not arise with even horizons, but the other cases may arise depending on the value of v_2 relative to other parameters. Figure 1 illustrates the ranges of v_2 corresponding to these cases.[1] The case in which the mall is built is highlighted in grey. The condition for the mall to be built and other properties illustrated in Figure 1 are established in Claim 6.

Claim 2 *In the two-seller game with $T = 2$, the mall is not built. In the subgame given participation, the mall is built if and only if*

$$\delta\pi^1_{B,1} - v_2 \geq 0 \tag{6}$$

where $\pi^1_{B,1}$ is the buyer's payoff in the one-seller game given participation. If the mall is built in the subgame, seller 2 sells in period 1 for $p^2_{2,2} = v_2$, and seller 1 sells in period 2 for $p^1_{1,1} = v_1$.

Proof. Suppose (6) holds, and consider period 2 first. If the buyer bargains with seller j in this period, she suggests a price $p^1_{j,1} = v_j$ such that the seller is indifferent between accepting and rejecting, and the seller accepts. Similarly, in period 1 the buyer suggests $p^2_{i,2} = v_i$, which seller i accepts. Therefore, if the buyer bargains with seller 2 first, her payoff is $\pi^2_{B,2} = \delta\pi^1_{B,1} - p^2_{2,2} = \delta(1 - v_1) - v_2$, which is positive because of (6). In contrast, if the buyer bargains with seller 1

[1] The figure does not include $T = 2$ because it mean little to discuss the cases. For example, if there is no agreement in period 1, the mall cannot be built because there are not enough periods to persuade both sellers.

first, her payoff is $\pi^{2\prime}_{B,2} = \delta(1-v_2) - v_1$, which is lower than $\pi^2_{B,2}$ because $\delta < 1$. Hence, the buyer prefers to bargain with seller 2 first.

If (6) is violated, both $\pi^2_{B,2}$ and $\pi^{2\prime}_{B,2}$ are negative, so the buyer does not initiate bargaining in period 1, so the mall is not built. ■

Claim 3 *In the two-seller game given participation with $T = 2t+2 \geq 4$, the first purchase cannot be in period 2.*

Proof. Suppose that there is no agreement in period 1 and that seller i sells at a price of $q^2_{i,2t+1}$ in period 2. In period 1, the seller would accept any price above $p^2_{i,2t+2} = H_{i,1} + \delta q^2_{i,2t+1}$, where the RHS is his payoff if he rejects $p^2_{i,2t+2}$. The buyer offers $p^2_{i,2t+2}$ in period 1 if it gives her a payoff no lower than waiting one period. That is,

$$\delta\pi^1_{B,2t+1} - p^2_{i,2t+2} \geq \delta[\delta\pi^1_{B,2t} - q^2_{i,2t+1}]. \tag{7}$$

where $\pi^1_{B,2t+1}$ and $\pi^1_{B,2t}$ are the buyer's payoffs in the one-seller games with $T = 2t+1$ and $T = 2t$ respectively. Substituting $p^2_{i,2t+2}$ into (7), we obtain $\delta\pi^1_{B,2t+1} - (H_{i,1} + \delta q^2_{i,2t+1}) \geq \delta[\delta\pi^1_{B,2t} - q^2_{i,2t+1}]$, or

$$\delta\pi^1_{B,2t+1} - v_i \geq \delta[\delta\pi^1_{B,2t} - v_i] \tag{8}$$

Recall that Claim 1 implies $\pi^1_{B,2t+1} > \pi^1_{B,2t}$, so (8) holds. Thus, there is an agreement in period 1, which contradicts with the assumption in the beginning of the proof. ■

Claim 4 *Suppose seller 2 sells in period 1 in the two-seller game given participation with $T = 2t$ for $t \geq 1$. Then, if the mall is built in the two-seller game with $T = 2t+2$, seller 2 also sells in period 1.*

Proof. If seller 2 sells in period 1 in the two-seller game with participation $T = 2t$, as in Figure 1, we have case "✓✓" or "✓×". If the mall is built in the two-seller game with $T = 2t+2$, we prove, in three steps, that the buyer first purchases from seller 2. Step I shows a useful preliminary result.

Step I. For $T = 2t$, seller 2's surplus is smaller than seller 1's, i.e.,

$$p^2_{2,T} - v_2 \leq \delta(p^1_{1,T-1} - v_1) \tag{9}$$

To see this, notice that if $T = 2$, (9) reduces to $v_2 - v_2 \leq \delta(v_1 - v_1)$, which is true. Suppose (9) holds for $T = 2t-2$. Then, under the assumption of Claim 4, there are two possibilities:

First, seller 2 sells in period 1 of $G(B, 2, 2t)$ and $G(2, B, 2t-1)$. Then,

$$\begin{aligned} p^2_{2,2t} - v_2 &= \delta(q^2_{2,2t-1} - v_2) & (10)\\ (\delta\pi^1_{B,2t-2} - v_2) - (q^2_{2,2t-1} - v_2) &= \delta[(\delta\pi^1_{B,2t-3} - v_2) - (p^2_{2,2t-2} - v_2)] & (11) \end{aligned}$$

Therefore,

$$
\begin{aligned}
p^2_{2,2t} - v_2 &= \delta(\delta\pi^1_{B,2t-2} - v_2) - \delta^2(\delta\pi^1_{B,2t-3} - v_2) + \delta^2(p^2_{2,2t-2} - v_2) \\
&\leq \delta(\delta\pi^1_{B,2t-2} - v_2) - \delta^2(\delta\pi^1_{B,2t-3} - v_2) + \delta^3(p^1_{1,2t-3} - v_1) \\
&= \delta^2(1 - v_1 - \pi^1_{1,2t-2}) - \delta v_2 - \delta^3(1 - v_1 - \pi^1_{1,2t-3}) + \delta^2 v_2 + \delta^3(p^1_{1,2t-3} - v_1) \\
&= -\delta^2\pi^1_{1,2t-2} + \delta^3\pi^1_{1,2t-3} + (\delta - \delta^2)(\delta(1 - v_1) - v_2) + \delta^3(p^1_{1,2t-3} - v_1) \\
&< (\delta - \delta^2)(\delta(1 - v_1) - v_2) + \delta^3(p^1_{1,2t-3} - v_1) \qquad (12)
\end{aligned}
$$

where the first inequality is from (9) for $T = 2t - 2$, and the last inequality is from $\delta\pi^1_{1,2t-3} < \pi^1_{1,2t-2}$ in Claim 1.

Second, suppose seller 2 sells in period 1 of $G(B, 2, 2t)$ but not in period 1 of $G(2, B, 2t-1)$. Then, $p^2_{2,2t} - v_2 = \delta^2(p^2_{2,2t-2} - v_2)$. Additionally, (9) for $T = 2t - 2$ implies $p^2_{2,2t} - v_2 = \delta^2(p^2_{2,2t-2} - v_2) \leq \delta^3(p^1_{1,2t-3} - v_1)$. Notice that $(\delta - \delta^2)(\delta(1 - v_1) - v_2) > 0$, so we also have (12). Hence, (12) holds in both possibilities.

In the one-seller game with $T = 2t - 1$, seller 1's surplus is the total surplus $1 - v_1$ minus the buyer's surplus, i.e.,

$$
\begin{aligned}
p^1_{1,2t-1} - v_1 &= (1 - v_1) - \pi^1_{B,2t-1} \\
&= (1 - v_1) - (1 - v_1) + \delta(1 - v_1) - \delta^2\pi^1_{B,2t-3} \\
&= \delta(1 - v_1) - \delta^2(1 - v_1) + \delta^2(p^1_{1,2t-3} - v_1)
\end{aligned}
$$

where the second equality is from iii) in Claim 1 and the last is due to the same reason of the first equality. Solving for $\delta^2(p^1_{1,2t-3} - v_1)$ from this equation and substituting it into (12), we obtain

$$
p^2_{2,2t} - v_2 < -v_2(\delta - \delta^2) + \delta(p^1_{1,2t-1} - v_1) < \delta(p^1_{1,2t-1} - v_1)
$$

which is (9) with $T = 2t$. Therefore, if (9) holds for $T = 2t - 2$, it also holds for $T = 2t$. Hence, by induction, the claim in Step I is true.

Step II. Characterize the necessary and sufficient condition for seller 1 to sell in period 1 of $G(B, 1, 2t+2)$. Consider two cases: First, seller 1 sells in period 1 of $G(B, 1, 2t+2)$ and $G(1, B, 2t+1)$. In period 1 of $G(1, B, 2t+1)$, the buyer accepts any price no higher than $q^2_{1,2t+1}$ such that $\delta\pi^{1\prime}_{B,2t} - q^2_{1,2t+1} = \delta(\delta\pi^1_{B,2t-1} - p^2_{2,2t})$, where $\pi^{1\prime}_{B,2t}$ is the buyer's payoff in the one-seller game with seller 1 with horizon $2t$. This equation means the buyer is indifferent between accepting the price or wait one more period. In addition, seller 1 offers such a price if $q^2_{1,2t+1} \geq H_{1,2} + \delta^2 p^1_{1,2t-1}$, which means his payoff from the offer is no lower than that from selling after seller 2. The two conditions above imply that seller 1 sells in period 1 of $G(1, B, 2t+1)$ if and only if $\delta\pi^{1\prime}_{B,2t} - \delta(\delta\pi^1_{B,2t-1} - p^2_{2,2t}) \geq H_{1,2} + \delta^2 p^1_{1,2t-1}$. Substituting $\pi^1_{B,2t-1} = 1 - p^1_{1,2t-1}$

and $H_{1,2} = (1-\delta)v_1(1+\delta)$ into this inequality, we can cancel $p^1_{1,2t-1}$ and obtain

$$\delta\pi^{1\prime}_{B,2t} - v_1 \geq \delta[\delta(1-v_1) - p^2_{2,2t}] \tag{13}$$

This means that the total surplus of the buyer and seller 1 in $G(1, B, 2t + 1)$ is no lower than the present value of their surplus in $G(B, 2, 2t)$. According to Claim 3, if seller 1 sells in period 1 of $G(1, B, 2t + 1)$, he must sell in period 1 of $G(B, 1, 2t + 2)$. Therefore, the first case arises if and only if (13) holds.

Second, consider the case in which seller 1 agrees in period 1 of $G(B, 1, 2t + 2)$ but not in period 1 of $G(1, B, 2t + 1)$. According to the derivation of (13), seller 1 does not sell in period 1 of $G(1, B, 2t + 1)$ if (13) is violated. Then, in period 1 of $G(B, 1, 2t + 2)$, seller 1 accepts any price higher than $p^2_{1,2t+2} = H_{1,3} + \delta^3 p^1_{1,2t-1}$, and the buyer offers such a price if $\delta\pi^{1\prime}_{B,2t+1} - p^2_{1,2t+2} \geq \delta^2(\delta\pi^1_{B,2t-1} - p^2_{2,2t})$. Substituting $p^2_{1,2t+2}$, $H_{1,3} = (1-\delta)v_1(1+\delta+\delta^2)$ and $\pi^1_{B,2t-1} = 1 - p^1_{1,2t-1}$ into this inequality, we obtain

$$\delta\pi^{1\prime}_{B,2t+1} - v_1 \geq \delta^2[\delta(1-v_1) - p^2_{2,2t}] \tag{14}$$

This means that the total surplus of the buyer and seller 1 in $G(B, 1, 2t + 2)$ is no lower than the present value of their surplus in $G(B, 2, 2t)$. Therefore, the second case arises if and only if (14) holds but (13) does not.

Recall that Claim 1 implies $\pi^{1\prime}_{B,2t+1} > \pi^{1\prime}_{B,2t}$. Therefore, the left hand side (LHS) of (13) is smaller than that of (14), while the right hand side (RHS) of (13) is larger. Hence, (13) implies (14). Then, combining the two cases above, we obtain that seller 1 sells in period 1 of $G(B, 1, 2t + 2)$ if and only if (14) holds.

Step III. Suppose seller 1 sells first when $T = 2t + 2$, then the buyer is better off by bargaining with seller 2 first instead.

If there is no agreement in the first two periods, the buyer purchases from seller 2 first according to the assumption in Claim 4. Therefore, we only need to consider the case in which seller 1 sells in period 1 or 2 when $T = 2t + 2$. Because Claim 3 already excludes the case in which seller 1 sells in period 2, it is sufficient to examine the case in which seller 1 sells in period 1.

By the same argument in Step II, the necessary and sufficient condition for seller 2 to sell in period 1 of $G(B, 2, 2t + 2)$ is

$$\delta\pi^1_{B,2t+1} - v_2 \geq \delta^2[\delta(1-v_1) - p^2_{2,2t}] \tag{15}$$

This means that the total surplus of the buyer and seller 2 in $G(B, 2, 2t + 2)$ is no lower than the present value of their surplus in $G(B, 2, 2t)$. If seller 1 sells in period 1 of $G(B, 1, 2t + 2)$, Step II implies that (14) holds. Recall that Claim 1 implies $\pi^1_{B,2t+1} > \pi^{1\prime}_{B,2t+1}$, so (14) implies (15). This means that if the buyer can purchase from seller 1 in period 1, so can she purchase from seller 2 in period 1.

In the remainder of Step III, we show that the buyer's payoff is higher if she purchases from seller 2 in period 1. We first derive the buyer's payoff if she purchases from seller 1 first, and then show that it is lower than her payoff if she purchases from seller 2 first. Specifically, consider one scenario in which seller 1 agrees in period 1 in $G(B,1,2t+2)$ and $G(1,B,2t+1)$. Then, analogues of (10) and (11) are $\delta\pi^{1\prime}_{B,2t} - q^2_{1,2t+1} = \delta\pi^2_{B,2t}$ and $p^2_{1,2t+2} - v_1 = \delta(q^2_{1,2t+1} - v_1)$. Therefore, the buyer's payoff is $\pi^{2\prime}_{B,2t+2} = \delta\pi^{1\prime}_{B,2t+1} - p^2_{1,2t+2} = (\delta\pi^{1\prime}_{B,2t+1} - v_1) - \delta(q^2_{1,2t+1} - v_1)$. Because the first purchase cannot in period 2 due to Claim 3, the above scenario arises if seller 1 is willing to offer $q^2_{1,2t+1}$ in period 1 of $G(1,B,2t+1)$, i.e., $q^2_{1,2t+1} - v_1 \geq \delta^2(p^1_{1,2t-1} - v_1)$, which is equivalent to

$$\delta\pi^{1\prime}_{B,2t} - v_1 - \delta\pi^2_{B,2t} \geq \delta^2(p^1_{1,2t-1} - v_1) \tag{16}$$

because of $\delta\pi^{1\prime}_{B,2t} - q^2_{1,2t+1} = \delta\pi^2_{B,2t}$.

If (16) does not hold, another scenario arises, where seller 1 agrees in period 1 in $G(B,1,2t+2)$ but not in period 1 of $G(1,B,2t+1)$. Then, seller 1 is indifferent between accepting and rejecting $p^2_{1,2t+2}$, which implies $p^2_{1,2t+2} - v_1 = \delta^3(p^1_{1,2t-1} - v_1)$. Therefore, the buyer's payoff is $\pi^2_{B,2t+2} = \delta\pi^{1\prime}_{B,2t+1} - p^2_{1,2t+2} = (\delta\pi^{1\prime}_{B,2t+1} - v_1) - \delta^3(p^1_{1,2t-1} - v_1)$. Because of Claim 3, there are no other scenarios besides the two above. Therefore, if the buyer purchases from seller 1 first, her payoff is

$$\begin{aligned}
\pi^{2\prime}_{B,2t+2} &= \begin{cases} (\delta\pi^{1\prime}_{B,2t+1} - v_1) - \delta(q^2_{1,2t+1} - v_1) & \text{if (16) holds} \\ (\delta\pi^{1\prime}_{B,2t+1} - v_1) - \delta^3(p^1_{1,2t-1} - v_1) & \text{otherwise} \end{cases} \\
&= \min\{\delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta(q^2_{1,2t+1} - v_1), \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta^3(p^1_{1,2t-1} - v_1)\}
\end{aligned}$$

which means the buyer's payoff is the smaller of the two payoffs derived above. Recall that $\delta\pi^{1\prime}_{B,2t} - q^2_{1,2t+1} = \delta\pi^2_{B,2t}$, so

$$\pi^{2\prime}_{B,2t+2} = \min\{\delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta(\delta\pi^{1\prime}_{B,2t} - v_1) + \delta^2\pi^2_{B,2t}, \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta^3(p^1_{1,2t-1} - v_1)\} \tag{17}$$

Next, we derive the buyer's payoff if she purchases from seller 2 first. By the same way to derive (16), we obtain that seller 2 sells in period 1 of $G(2,B,2t+1)$ if and only if

$$q^2_{2,2t+1} - v_2 \geq \delta(p^2_{2,2t} - v_2) \tag{18}$$

where the RHS is seller 2's surplus from selling after one period of delay. In contrast, the RHS of (16) is seller 1's surplus from selling after two period of delay. Moreover, we show in the beginning of Step III that seller 2 sells in period 1 of $G(B,2,2t+1)$ whether (18) holds. Therefore, case "✓✓" arises under (18) and case "✓×" arises otherwise. Following the same analysis for (17), we obtain the buyer's payoff as

$$\begin{aligned}
\pi^2_{B,2t+2} &= \begin{cases} (\delta\pi^1_{B,2t+1} - v_2) - \delta(q^2_{2,2t+1} - v_2) & \text{if (18) holds} \\ (\delta\pi^1_{B,2t+1} - v_2) - \delta^2(p^2_{2,2t} - v_2) & \text{otherwise} \end{cases} \\
&= \min\{(\delta\pi^1_{B,2t+1} - v_2) - \delta(q^2_{2,2t+1} - v_2), (\delta\pi^1_{B,2t+1} - v_2) - \delta^2(p^2_{2,2t} - v_2)\}
\end{aligned}$$

Because $\delta\pi^1_{B,2t} - q^2_{2,2t+1} = \delta\pi^2_{B,2t}$, we obtain

$$\pi^2_{B,2t+2} = \min\{\delta\pi^1_{B,2t+1} - v_2 - \delta(\delta\pi^1_{B,2t} - v_2) + \delta^2\pi^2_{B,2t}, \delta\pi^1_{B,2t+1} - v_2 - \delta^2(p^2_{2,2t} - v_2)\} \quad (19)$$

Given the above expressions of $\pi^2_{B,2t+2}$ and $\pi^{2\prime}_{B,2t+2}$, we prove below that $\pi^2_{B,2t+2} > \pi^{2\prime}_{B,2t+2}$. Notice that $\pi^2_{B,2t+2}$ is the minimum of two terms. Therefore, to show $\pi^2_{B,2t+2} > \pi^{2\prime}_{B,2t+2}$, it is sufficient to verify that, whichever term in (19) $\pi^2_{B,2t+2}$ equals to, it is larger than $\pi^{2\prime}_{B,2t+2}$. Specifically,

$$\begin{aligned}
\text{The second term in (19)} &= \delta\alpha^1_{1,2t+1}(1-v_1) - v_2 - \delta^2(p^2_{2,2t} - v_2) \\
&> \delta\alpha^1_{1,2t+1}(1-v_2) - v_1 - \delta^2(p^2_{2,2t} - v_2) \\
&= \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta^2(p^2_{2,2t} - v_2) \\
&\geq \text{The second term in (17)} \\
&\geq \pi^{2\prime}_{B,2t+2}
\end{aligned}$$

where the first inequality is from $\alpha^1_{1,2t+1} < 1$ for $t \geq 1$ in ii) of Claim 1, and the third inequality from (9).

As a result it remains to show that whenever $\pi^2_{B,2t+2}$ equals the first term in (19), it exceeds $\pi^{2\prime}_{B,2t+2}$. Suppose otherwise, then, because both $\pi^2_{B,2t+2}$ and $\pi^{2\prime}_{B,2t+2}$ are continuous in v_1 and v_2, there must be some v_1 and v_2 such that the two are equal, i.e.,

$$\delta\pi^1_{B,2t+1} - v_2 - \delta(\delta\pi^1_{B,2t} - v_2) + \delta^2\pi^2_{B,2t} = \pi^{2\prime}_{B,2t+2} \quad (20)$$

from which we construct a contradiction below.

There are two possible scenarios. In the first scenario, (16) is voilated. Then, $\pi^{2\prime}_{B,2t+2}$ equals the second term in (17), and (20) becomes

$$\delta\pi^1_{B,2t+1} - v_2 - \delta(\delta\pi^1_{B,2t} - v_2) + \delta^2\pi^2_{B,2t} = \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta^3(p^1_{1,2t-1} - v_1)$$

Equation (16) is voilated, i.e., $\delta\pi^2_{B,2t} > \delta\pi^{1\prime}_{B,2t} - v_1 - \delta^2(p^1_{1,2t-1} - v_1)$, which, combined with the above equation, implies

$$\delta\pi^1_{B,2t+1} - v_2 - \delta(\delta\pi^1_{B,2t} - v_2) < \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta(\delta\pi^{1\prime}_{B,2t} - v_1) \quad (21)$$

Recall that $\pi^1_{B,T} = \alpha^1_{B,T}(1-v_1)$ and $\pi^{1\prime}_{B,T} = \alpha^1_{B,T}(1-v_2)$, so the above inequality is equivalent to

$$\delta\alpha^1_{B,2t+1} - \delta^2\alpha^1_{B,2t} > 1 - \delta \quad (22)$$

where $v_1 - v_2$ multiplies to both sides so is cancelled. Recall that in (19), $\pi^2_{B,2t+2}$ equals the

LHS of (20) if (18) holds. We can also rewrite (18) as

$$\delta\alpha^1_{B,2t} - \delta^2\alpha^1_{B,2t-1} > (1-\delta)v_2/(1-v_1)$$

which combined with (22) gives

$$\delta\alpha^1_{B,2t+1} - \delta^3\alpha^1_{B,2t-1} > 1-\delta+(1-\delta)\delta v_2/(1-v_1) \tag{23}$$

Recall that $\alpha^1_{1,T+2} = (1-\delta)+\delta^2\alpha^1_{1,T}$ in Claim 1. Substituting this equation into (23) and rearranging terms, we obtain $0 > 1-\delta+v_2/(1-v_1)$, which cannot be true because the RHS is positive.

In the second scenario, (16) holds. Then, $\pi^{2\prime}_{B,2t+2}$ equals the first term in (17), and (20) becomes

$$\delta\pi^1_{B,2t+1} - v_2 - \delta(\delta\pi^1_{B,2t} - v_2) + \delta^2\pi^2_{B,2t} = \delta\pi^{1\prime}_{B,2t+1} - v_1 - \delta(\delta\pi^{1\prime}_{B,2t} - v_1) + \delta^2\pi^2_{B,2t}$$

which is equivalent to (21) with equality. Therefore, we also have (22) with equality. Following the same argument in the first scenario, we obtain $0 \geq 1-\delta+v_2/(1-v_1)$, which cannot be true either because the RHS is positive.

As a result, $\pi^{2\prime}_{B,2t+2} < \pi^2_{B,2t+2}$, which means the buyer prefers to bargain with seller 2 first. ■

For any even $T > 2$, Figure 1 shows that case "××" arises if and only if v_2 is above a critical value, which is invariant with T. The claim below shows that the critical value is $\delta(1-v_1)/(1+\delta) \equiv \delta\pi^1_{B,\infty}$. Recall that case "××" means there is no agreement in the first two periods with seller 2, and the claim below shows a stronger result that there is no agreement in the first two periods with seller 1 either.

Claim 5 *For any even horizon $T = 2t \geq 4$, there is no purchase in periods 1 and 2 in the two-seller game given participation if and only if*

$$\delta \lim_{t\to\infty} \pi^1_{B,2t-1} < v_2 \tag{24}$$

Proof. From ii) of Claim 1, $\lim_{t\to\infty}\pi^1_{B,2t-1} = (1-v_1)/(1+\delta)$. The rest of the proof has four steps. First, suppose the mall is not built in $G(B,2,2)$, which means $v_2 > \delta\pi^1_{B,1}$. We show below that the mall is not built in $G(B,2,2t)$ for $t > 1$ either. To see this, suppose the mall is not built if the buyer rejects in period 1 of $G(2,B,2t-1)$. Then, the buyer is indifferent between accepting and rejecting $q^2_{2,2t-1}$ in period 1 if $\delta\pi^1_{B,2t-2} - q^2_{2,2t-1} = 0$. Seller 2 offers such a price if $q^2_{2,2t-1} \geq v_2$. Therefore, there is no agreement in period 1 of $G(2,B,2t-1)$ if $\delta\pi^1_{B,2t-2} < v_2$. Similarly, there is no agreement in period 1 of $G(B,2,2t)$ if $\delta\pi^1_{B,2t-1} < v_2$. Recall that Claim 1 implies $\pi^1_{B,1} > \pi^1_{B,2t-1} > \pi^1_{B,2t-2}$, so if $v_2 > \delta\pi^1_{B,1}$, there is no agreement in periods 1 and 2 of $G(B,2,2t)$, and the mall is not built.

Second, suppose the mall is built in $G(B,2,2)$, which means $v_2 \geq \delta\pi^1_{B,1}$. We characterize below the condition for no purchase in periods 1 and 2 of $G(B,2,2t)$. According to Claim 4, we only need to find the condition for seller 2 not to sell in the first two periods.

Consider $G(2,B,2t-1)$. Suppose the mall is eventually built if the buyer rejects in period 1. Then, denote the number of periods left after the first purchase as $2t'$ with $1 \leq t' < t$.[2] Claim 3 implies the first seller is seller 2. Then, the buyer in $G(2,B,2t-1)$ is indifferent between accepting and rejecting q^2_{2t-1} if $\delta\pi^1_{B,2t-2} - q^2_{2t-1} = \delta^{2(t-t')-1}(\delta\pi^1_{B,2t'-1} - p^2_{2,2t'})$, and seller 2 offers $q^2_{2,2t-1}$ if $q^2_{2t-1} - v_2 \geq \delta^{2(t-t')-1}(p^2_{2,2t'} - v_2)$. Therefore, no agreement in period 1 of $G(2,B,2t-1)$ if

$$\delta\pi^1_{B,2t-2} - \delta^{2(t-t')-1}(\delta\pi^1_{B,2t'-1} - p^2_{2,2t'}) < \delta^{2(t-t')-1}(p^2_{2,2t'} - v_2)$$

or

$$\delta\pi^1_{B,2t-2} - v_2 < \delta^{2(t-t')-1}(\delta\pi^1_{B,2t'-1} - v_2) \tag{25}$$

Consider $G(B,2,2t)$. Under (25), seller 2 is indifferent between accepting and rejecting $p^2_{2,2t}$ if $p^2_{2,2t} - v_2 > \delta^{2(t-t')}(p^2_{2,2t'} - v_2)$, and the buyer offers $p^2_{2,2t}$ if $\delta\pi^1_{B,2t-1} - p^2_{2,2t} \geq \delta^{2(t-t')}(\delta\pi^1_{B,2t'-1} - p^2_{2,2t'})$. Therefore, no agreement in period 1 of $G(B,2,2t)$ if

$$\delta\pi^1_{B,2t-1} - v_2 - \delta^{2(t-t')}(p^2_{2,2t'} - v_2) < \delta^{2(t-t')}(\delta\pi^1_{B,2t'-1} - p^2_{2,2t'})$$

or

$$\delta\pi^1_{B,2t-1} - v_2 < \delta^{2(t-t')}(\delta\pi^1_{B,2t'-1} - v_2) \tag{26}$$

which means the total surplus of seller 2 and the buyer is less than the present value of their total surplus if the first agreement is delayed until $2t'$ periods are left. Hence, no agreement in period 1 and 2 of $G(B,2,2t)$ if (25) and (26) hold.

Third, we verify that (26) implies (25). To see this, rewrite (25) as

$$\delta(\delta\pi^1_{B,2t-2} - v_2) < \delta^{2(t-t')}(\delta\pi^1_{B,2t'-1} - v_2) \tag{27}$$

Notice that the RHS is the same as that in (26), so it is sufficient to show that the LHS of (26) is larger than that of (27), i.e., $\delta\pi^1_{B,2t-1} - v_2 > \delta(\delta\pi^1_{B,2t-2} - v_2)$. Recall that Claim 1 implies $\pi^1_{B,2t-1} > \pi^1_{B,2t-2}$, so $\delta\pi^1_{B,2t-1} - v_2 > \delta\pi^1_{B,2t-2} - v_2 > \delta(\delta\pi^1_{B,2t-2} - v_2)$.

Fourth, we prove Claim 5. Notice that the two steps above imply no agreement in the first two periods of $G(B,2,2t)$ if and only if (26) holds. Rewrite (26) as

$$\frac{\delta\pi^1_{B,2t-1} - \delta^{2(t-t')}\delta\pi^1_{B,2t'-1}}{1 - \delta^{2(t-t')}} < v_2 \tag{28}$$

Next, we show that (28) is equivalent to (24). Recall that in iii) of Claim 1, we have $\pi^1_{B,2t-1} = (1-v_1)(1-\delta) + \delta^2\pi^1_{B,2t-3}$, repetition of which implies $\pi^1_{B,2t-1} = (1-v_1)(1-\delta+\delta^2 - ... - \delta^{2(t-t')-1}) + \delta^{2(t-t')}\pi^1_{B,2t'-1}$. Substituting the above expression into (28), we can rewrite (28)

[2] It turns out that $t' = 1$. See Figure 1.

as (24). Notice that (24) implies $\delta \lim_{t\to\infty} \pi^{1\prime}_{B,2t-1} < v_1$, so there is no agreement in period 1 of $G(B, 1, 2t)$ and $G(1, B, 2t-1)$. Hence, whichever seller the buyer bargains with, there is no agreement in the first two periods. ■

Note that condition (24) does not depend on T, and inequality (26) gives an intuition: On the one hand, if the buyer and seller 2 have an agreement in the first two periods, a longer horizon increases the horizon of the one-seller game after the agreement. According to Claim 1, this decreases $\pi^1_{B,2t-1}$, and reduces the buyer and seller 2's total surplus from the mall. Therefore, the agreement between them is less "attractive". On the other hand, if the buyer and seller 2 have no agreement in the first two periods, a longer horizon increases the delay before the next agreement. Therefore, the agreement between the buyer and seller 2 becomes more "attractive". These two effects cancel each other, so (24) is independent of the horizon.

Next, we use the above claims to prove the proposition below, which is Proposition 1 in the main paper.

Proposition 1 *For any $N \geq 2$ and any $T \geq 2$, the N-seller game with horizon T has a unique equilibrium outcome. Moreover, if the mall is built in the outcome, in the first N periods the buyer purchases from the N sellers in the order of increasing size.*

Proof of Proposition 1 for $N = 2$ and $T = 2t$.

Consider the two-seller game with sellers 1 and 2. We start with $T = 2$, then Claim 2 implies Proposition 1 for $T = 2$. In addition, Claim 2 implies that if $T = 2$, the mall is built in the subgame given participation if and only if (6) holds. First, consider the case in which the mall is not built in the subgame given participation when $T = 2$. This case arises if (6) is violated. Claim 5 implies that in the subgame given participation with $T = 4$, the mall is not built if and only if (24) holds. We can verify that (24) holds if (6) is violated. Therefore, the mall is not built for $T = 4$, so Proposition 1 holds for $T = 4$.

Second, consider the other case in which the mall is built in the subgame given participation when $T = 2$. Then, Claim 2 implies that seller 2 sells first in the subgame. Claim 4 implies that if the mall is built when $T = 4$, the buyer also purchases from seller 2 in period 1. Hence, the two-seller game has a unique equilibrium outcome, and Proposition 1 holds for $T = 4$.

So far, we use Proposition 1 for $T = 2$ to prove it for $T = 4$. Next, suppose the proposition is true for $T = 2t \geq 4$, and we prove the proposition for $T = 2t + 2$. Suppose $T = 2t + 2$ and (24) holds, then Claim 5 implies that there is no purchase in the first two periods, and in resulting subgame of $2t$ periods, Proposition 1 holds by assumption. Thus, the proposition holds for $T = 2t + 2$.

Consider $T = 2t + 2$ and suppose (24) does not hold. Then, in the game with horizon $2t$ given participation, Claim 5 implies that there is a purchase in the first two periods. Moreover, Claim 3 implies that the purchase is in period 1, and Step III for Claim 4 implies that the first seller is seller 2. Therefore, seller 2 sells in period 1 in the game given participation when the horizon is $2t$. Then, Claim 4 implies that if the mall is built with horizon $2t + 2$, seller 2 also sells in period 1. Hence, Proposition 1 holds for $T = 2t + 2$ in this case as well. Therefore, Proposition 1 holds in the two-seller game with any even horizon. ■

In the above proof, we prove Proposition 1 for $N = 2$ and $T = 2t$ without specifying the condition for the mall is built. To complete the analysis for even horizons, the following result characterizes the evolution of the three cases and the condition for the mall to be built. These properties are used to prove Proposition 1 for $N \geq 3$.

Claim 6 *In the two-seller game given participation with horizon $T = 2t + 2 \geq 4$,*

i) case "✓✓" arises if

$$\delta\pi^1_{B,T-2} - v_2 \geq \delta(\delta\pi^1_{B,T-3} - v_2) \tag{29}$$

ii) case "✓×" arises if neither (24) nor (29) holds

iii) case "××" arises if (24) holds

iv) "✓✓" for $T - 2$ implies "✓✓" for T

v) the mall is built if and only if (29) holds with a strict inequality

Proof. First, we discuss the evolution of the three possible cases for even horizons. According to Claim 5, case "××" arises for $T = 2t \geq 4$ if (24) holds. Suppose (24) does not hold, then case "✓✓" or "✓×" arises. According to the proof of Claim 4, case "✓✓" arises for $T = 4$ if and only if (18) holds for $t = 1$. Using $\delta\pi^1_{B,2t} - q^2_{2,2t+1} = \delta\pi^2_{B,2t}$ and $\delta\pi^1_{B,2t-1} - p^2_{2,2t} = \pi^2_{B,2t}$, we can rewrite (18) as (29), where $\pi^1_{B,1} = 1 - v_1$. It is equivalent to $\delta\pi^1_{B,2} - \delta^2\pi^1_{B,1} \geq (1 - \delta)v_2$. According to Claim 1, $\pi^1_{B,2} < \pi^1_{B,1}$, so $\delta\pi^1_{B,2} - \delta^2\pi^1_{B,1} \leq \delta\pi^1_{B,2} - \delta^2\pi^1_{B,2}$. Therefore, if (29) holds for $T = 2$, (24) does not hold. As a result, case "✓✓" arises for $T = 4$ if (29) holds. If neither (24) nor (29) holds, then case "✓×" arises. Similarly,

i) case "✓✓" arises for $T = 2t + 2$ if (29) holds;

ii) case "✓×" arises if neither (24) nor (29) holds, and

iii) case "××" arises if (24) holds.

Note that (29) implies that the critical value of v_2 dividing the cases "✓×" and "✓✓" is $(\delta\pi^1_{B,2t} - \delta^2\pi^1_{B,2t-1})/(1 - \delta)$. According to Claim 1, $\pi^1_{B,2} < \pi^1_{B,4} < ... < \lim_{t\to\infty}\pi^1_{B,2t} = \lim_{t\to\infty}\pi^1_{B,2t-1} < ... < \pi^1_{B,3} < \pi^1_{B,1}$. Therefore, the critical value increases in t. As a result, iv) case "✓✓" for $T = 2t$ implies case "✓✓" for $T = 2t + 2$.

Second, we derive the condition for the mall to be built. For $T = 2$, Claim 2 implies that the mall is not built. For $T = 2t + 2 \geq 4$, the mall is not built in case "××". To see this, notice that the case arises if (24) holds, then Claim 5 implies that there is no agreement until there are two periods left. If the mall is built in the subgame with two periods, both sellers receive a zero surplus according to Claim 2. If the mall is not built in the subgame with two periods, every player receives a zero surplus. Therefore, if (24) holds, the sellers receive a zero surplus if they participate. Hence, the mall is not built.

For $T = 2t + 2 \geq 4$, the mall is not built in case "✓×". The property iv) above implies that if "✓×" arises for $T = 2t + 2$, it arises for $T = 4, 6, ..., 2t$. Recall that the buyer offers a price such that seller 2 is indifferent between accepting and rejecting, which in case "✓×" means that seller 2's surplus is the same for $T = 2t+2$ and $T = 2t$. Repeating this analysis, we obtain that seller 2's surplus is the same as $T = 2$, which is zero. Hence, if seller 2 participates, his surplus is zero, so the mall is not built if case "✓×" for $T = 2t+2$.

For $T = 2t+2 \geq 4$, the mall is built in case "✓✓" if (29) holds with a strict inequality. It is sufficient to verify that every player receives a positive surplus. In the proof of Claim 4, if (29) holds with equality, seller 2 is indifferent between offering $q^2_{2,2t+1}$ and waiting one period in $G(2, B, 2t+1)$. If (29) holds with a strict inequality, seller 2 strictly prefers offering $q^2_{2,2t+1}$ than waiting one period. Because seller 2's surplus is nonnegative by waiting one period, her surplus must be positive in $G(2, B, 2t+1)$. In addition, seller 2's surplus in $G(B, 2, 2t+2)$ cannot be lower than that in its subgame $G(2, B, 2t+1)$, so seller 2's surplus is positive if (29) holds with a strict inequality. The buyer's surplus in $G(2, B, 2t+1)$ is nonnegative, and Claim 3 shows that her payoff is even higher in $G(B, 2, 2t+2)$. As a result, the buyer also has a positive surplus. Finally, Claim 1 implies that $\alpha^1_{1,T} \in (0, 1)$, so seller 1's surplus is $\pi^1_{1,2t+1} = (1 - v_1)\alpha^1_{1,2t+1} > 0$. Hence, we verify that every player's surplus is positive, so the mall is built. ■

As a result of the above claim, the mall is built only in case "✓✓", which is highlighted in grey in Figure 1. So far we have discussed the two-seller game with an even horizon. Next, we consider odd horizons. Claim 7 proves for $T = 3$ and demonstrates three cases: "✓✓", "×✓", "××". Figure 2 illustrates these cases. In contrast to Figure 2, case "✓×" does not arise with odd horizons. Next, suppose Proposition 1 is true for horizon $T = 2t-1$ and consider $T = 2t+1$. As in the figure, if we have "✓✓" when $T = 2t-1$, the mall is built. According to Claim 8, we also have "✓✓" when $T = 2t+1$, and seller 2 sells first. If we have "×✓" or "××" when $T = 2t-1$, the mall is not built. However, the mall may be built when $T = 2t+1$, and if it is, Claims 8 shows seller 2 sells first.

Claim 7 *In the two-seller game with $T = 3$, the mall is built if and only if*

$$\delta\pi^1_{B,T-1} - v_2 \quad > \quad \delta(\delta\pi^1_{B,T-2} - v_2) \tag{30}$$

If the mall is built, the buyer purchases from seller 2 in period 1.

Proof. First, case "✓✓" arises if

$$\delta\pi^1_{B,T-1} - v_2 \quad \geq \quad \delta(\delta\pi^1_{B,T-2} - v_2) \tag{31}$$

which is (30) with a weak inequality. In the final period, if neither seller has agreed, the mall is not built. Suppose that only seller 1 has not sold by the final period. If the buyer offers in the last period, she would suggest $p^1_{1,1} = v_1$ to seller 1. If seller 1 offers in the final period, he would suggest $q^1_{1,1} = 1$. In either case, the offering player extracts all the surpluses.

Let us move backwards to the second period. Suppose that neither seller has agreed in this period. Then, in $G(2, B, 2)$, the buyer is indifferent between accepting and rejecting if seller 2 offers $q^2_{2,2}$ such that $\delta\pi^1_{B,1} - q^2_{2,2} = 0$, so $q^2_{2,2} = \delta\pi^1_{B,1}$. The seller 2 offers such a price if $q^2_{2,2} \geq v_2$. Therefore, there is agreement in period 1 of $G(2, B, 2)$ if and only if $\delta\pi^1_{B,1} \geq v_2$. We claim that (31) implies $\delta\pi^1_{B,1} \geq v_2$, so seller 2 sells in period 1 of $G(2, B, 2)$. To see this, rewrite (31) as $\delta(\pi^1_{B,2} - \delta\pi^1_{B,1})/(1-\delta) \geq v_2$. Combine this inequality with $\pi^1_{B,2} < \pi^1_{B,1}$ from Claim 1, we obtain $\delta\pi^1_{B,1} = \delta(\pi^1_{B,1} - \delta\pi^1_{B,1})/(1-\delta) > \delta(\pi^1_{B,2} - \delta\pi^1_{B,1})/(1-\delta) \geq v_2$, so (31) implies $\delta\pi^1_{B,1} \geq v_2$.

Consider the first period. If the buyer bargains with seller 2 in period 1, the seller is indifferent between accepting and rejecting if the buyer suggests $p^2_{2,3} = H_{2,1} + \delta q^2_{2,2}$. Recall that $H_{2,1} = (1-\delta)v_2$ and $q^2_{2,2} = \delta\pi^1_{B,1}$, so $p^2_{2,3} = (1-\delta)v_2 + \delta^2\pi^1_{B,1}$. Moreover, the buyer offers $p^2_{2,3}$ if her payoff is no lower than 0, which is her payoff in $G(2, B, 2)$. Notice $\pi^2_{B,3} = \delta\pi^1_{B,2} - p^2_{2,3} = \delta\pi^1_{B,2} - v_2 - \delta(\delta\pi^1_{B,1} - v_2) \geq 0$, where the inequality is from (31). Therefore, if the buyer bargains with seller 2 first, case "✓✓" arises, and the seller sells in period 1 and the mall is built.

Second, we show that if (30) holds, the mall is built and the buyer purchases from seller 2 in period 1. We show above that if the buyer bargains with seller 2 in period 1, the mall is built. Recall that the resulting payoff for the buyer $\pi^2_{B,3} \geq 0$ if (31) holds. Thus, if (30) holds, $\pi^2_{B,3} > 0$, so the buyer chooses to participate.

Next, we show that once the buyer participates, she does not purchase from seller 1 first. Suppose the buyer purchases from seller 1 first. Then, her payoff is either 0 if seller 1 sells in period 2, or $\pi^{2\prime}_{B,3} = \delta\pi^{1\prime}_{B,2} - p^2_{1,3}$ if seller 1 sells in period 1. We verify below that the buyer's payoff in either case is lower than $\pi^2_{B,3}$. Specifically, recall that $\pi^2_{B,3} > 0$ under (30), so it remains to show $\pi^2_{B,3} > \delta\pi^{2\prime}_{B,3}$. Recall that $\pi^2_{B,3} = \pi^{1\prime}_{B,2} - v_2 - \delta(\delta\pi^1_{B,1} - v_2)$, similarly, we have $\pi^{2\prime}_{B,3} = \delta\pi^{1\prime}_{B,2} - v_1 - \delta(\delta\pi^{1\prime}_{B,1} - v_1)$. Therefore, $\pi^2_{B,3} > \delta\pi^{2\prime}_{B,3}$ is equivalent to $\delta\pi^1_{B,2} - v_2 - \delta(\delta\pi^1_{B,1} - v_2) > \delta\pi^{1\prime}_{B,2} - v_1 - \delta(\delta\pi^{1\prime}_{B,1} - v_1)$, or

$$(\pi^1_{B,2} - \delta\pi^1_{B,1})\delta/(1+\delta) - v_2 > (\pi^{1\prime}_{B,2} - \delta\pi^{1\prime}_{B,1})\delta/(1+\delta) - v_1$$

Recall that Claim 1 implies $\pi^1_{B,2} = \alpha^1_{1,2}(1-v_1)$ and $\pi^1_{B,1} = \alpha^1_{1,1}(1-v_1)$. Therefore, we can rewrite LHS of the above inequality as $(1-v_1)(\alpha^1_{1,2} - \delta\alpha^1_{1,1})\delta/(1+\delta) - v_2$. Similarly, the RHS of the inequality can be rewritten as $(1-v_2)(\alpha^1_{1,2} - \delta\alpha^1_{1,1})\delta/(1+\delta) - v_1$. According to Claim 1, $\alpha^1_{1,2} < 1$ and $\alpha^1_{1,1} > 0$, so $(\alpha^1_{1,2} - \delta\alpha^1_{1,1})\delta/(1+\delta) < 1$, so $(1-v_1)(\alpha^1_{1,2} - \delta\alpha^1_{1,1})\delta/(1+\delta) - v_2 > (1-v_2)(\alpha^1_{1,2} - \delta\alpha^1_{1,1})\delta/(1+\delta) - v_1$, which is equivalent to $\pi^2_{B,3} > \delta\pi^{2\prime}_{B,3}$.

Third, if $\delta\pi^1_{B,1} \geq v_2$ holds but (31) does not, case "×✓" arises, and the mall is not built. Suppose the buyer bargains with seller 2 first. If $\delta\pi^1_{B,1} \geq v_2$ holds but (31) does not, the above analysis implies that there is no agreement in period 1 of $G(B, 2, 3)$, but seller 2 sells in period 1 of $G(2, B, 2)$. Hence, case "×✓" arises.

Next, we show that in this case, the mall is not built. In subgame $G(2, B, 2)$, seller 2 offers such that the buyer's payoff is zero. Therefore, in the subgame after the buyer chooses seller 2 to bargain with first, the buyer's payoff is zero. Suppose the buyer bargains with seller 1 first. As above, seller 1 sells in period 1 of $G(1, B, 2)$ if and only if $\delta\pi^{1\prime}_{B,1} \geq v_1$. Moreover, seller 1 sells in period 1 of $G(B, 1, 3)$ if and only if $\delta\pi^{1\prime}_{B,T-1} - v_1 \geq \delta(\delta\pi^{1\prime}_{B,T-2} - v_1)$ or $\pi^{2\prime}_{B,3} \geq 0$. We show above that (31) implies $\delta\pi^1_{B,1} \geq v_2$. By the same argument, $\delta\pi^{1\prime}_{B,T-1} - v_1 \geq \delta(\delta\pi^{1\prime}_{B,T-2} - v_1)$

implies $\delta\pi^{1\prime}_{B,1} \geq v_1$. As a result, $\pi^{2\prime}_{B,3} \geq 0$ implies $\delta\pi^{1\prime}_{B,1} \geq v_1$. Hence, seller 1 sells in period 1 if and only if $\pi^{2\prime}_{B,3} \geq 0$. Notice that if (31) does not hold, neither does (30). Recall that the inverse of (30) is equivalent to $\pi^2_{B,3} \leq 0$ and that $\pi^2_{B,3} > \pi^{2\prime}_{B,3}$, so $\pi^{2\prime}_{B,3} < 0$. This means seller 1 does not sell in period 1 under (30). Therefore, in subgame $G(B, 2, 3)$, either the seller 1 sells in period 2 or the mall is not built. In either case, the buyer's payoff is zero so she does not participate the bargaining game. Hence, the mall is not built if the buyer bargains with seller 1 first.

Fourth, case "××" arises if neither $\delta\pi^1_{B,1} \geq v_2$ nor (31) holds. If the buyer bargains with seller 2 first, the above analysis implies there is no agreement in periods 1 and 2, with only one period left, the mall cannot be built. Suppose the buyer bargains with seller 1 first. If $\delta\pi^1_{B,1} \geq v_2$ does not hold, neither does $\delta\pi^{1\prime}_{B,1} \geq v_1$. In addition, recall that if (31) does not hold, $\pi^{2\prime}_{B,3} > 0$ does not hold. Hence, seller 1 does not sell in periods 1 and 2, so the mall is not built either. ■

Consider sellers such that case "×✓" or "××" arises for $T = 2t - 1$, the result below studies what happens for the same sellers if $T = 2t + 1$.[3]

Claim 8 *Suppose (30) does not hold for $T = 2t - 1$. Then, in the game with $T = 2t + 1$, if (30) holds for $T = 2t + 1$, seller 2 sells in period 1 and the mall is built. Otherwise, the mall is not built because it would result in a zero payoff for the buyer.*

Proof. We prove by induction. Consider $t = 2$ and suppose (30) does not hold for $T = 2t - 1$. Then, Claim 7 implies that the mall is not built, and it can be case "×✓", "××", or the boundary of case "✓✓" when (30) is violated with an equality. In each of these cases, the buyer's payoff is zero. Notice that if no agreements in periods 1 and 2 in the two-seller game with $T = 3$, the mall is not built and the buyer's payoff is also zero. Because of this similarity, the rest of the proof is similar to that of Claim 7.

First, case "✓✓" arises if (31) holds for $T = 2t + 1 = 5$. Suppose the buyer bargains with seller 2 first. Then, in period 1 of $G(2, B, 2t)$, seller 2 offers $q^2_{2,2t}$ such that the buyer is indifferent between accepting and rejecting. That is,

$$\delta\pi^1_{B,2t-1} - q^2_{2,2t} = 0. \tag{32}$$

Seller 2 offers $q^2_{2,2t}$ if $q^2_{2,2t} \geq v_2$. Solving $q^2_{2,2t}$ from (32) and substituting it into the inequality above, we get $\delta\pi^1_{B,2t-1} \geq v_2$. In the proof of Claim 7, we show (31) implies $\delta\pi^1_{B,1} \geq v_2$. By the same argument, (31) also implies $\delta\pi^1_{B,2t-1} \geq v_2$. As a result, seller 2 sells in period 1 of $G(2, B, 2t)$ if (31) holds.

In period 1 of $G(B, 2, 2t+1)$, seller 2 accepts any price no lower than $p^2_{2,2t+1} = H_{2,1} + \delta q^2_{2,2t}$, and the buyer offers $p^2_{2,2t+1}$ if

$$\delta\pi^1_{B,2t} - p^2_{2,2t+1} \geq \delta\left[\delta\pi^1_{B,2t-1} - q^2_{2,2t}\right]. \tag{33}$$

[3]Claim 8 also discusses the case in which (31) holds with an equality. This is a special case of "✓✓".

Substituting $p^2_{2,2t+1}$ into the inequality above, we get (31). Therefore, under (31), seller 2 sells in period 1 of $G(B, 2, 2t+1)$ and $G(2, B, 2t)$, which means "✓✓" arises.

Second, we show that if (30) holds for $T = 2t+1$, the mall is built and the buyer purchases from seller 2 in period 1. In contrast, suppose the buyer purchases from seller 1 first. Then, her payoff is either 0 if seller 1 sells in period 2, or $\pi^{2\prime}_{B,2t+1} = \delta\pi^{1\prime}_{B,2t} - p^2_{1,2t+1}$ if seller 1 sells in period 1. In the proof of Claim 7, we verify $\pi^2_{B,3} > \pi^{2\prime}_{B,3}$. Similarly, we can verify $\pi^2_{B,2t+1} > \pi^{2\prime}_{B,2t+1}$.[4] Therefore, the buyer prefers to bargain with seller 2 first.

Third, case "×✓" arises if $\delta\pi^1_{B,2t-1} \geq v_2$ holds but (31) does not. By the same argument in the third step in the proof of Claim 7, we can show that the buyer receives a zero payoff whichever seller she bargains with first. Hence, she does not participate the bargaining game and the mall is not built.

Fourth, case "××" arises if neither $\delta\pi^1_{B,2t-1} \geq v_2$ nor (31) holds. By the same argument in the fourth step in the proof of Claim 7, the mall cannot be built whichever seller the buyer bargains with first. Hence, the mall is not built either.

So far, we prove the claim for $t = 2$. Suppose Claim 8 is true for any $t' \geq 2$, it remains to show the claim for $t = t'+1$. Consider $t = t'+1$ and suppose (30) does not hold for $T = 2t'+1$, then Claim 8 for $t = t'$ implies that the mall is not built because it would result in a zero payoff for the buyer. Then, repeating the analysis above, we can prove that in the game with $T = 2t'+3$, if (30) holds for $T = 2t'+3$, seller 2 sells in period 1 and the mall is built. If (30) does not hold for $T = 2t'+3$, the mall is not built because it would result in a zero payoff for the buyer. Thus, Claim 8 holds for $t = t'+1$ as well. ■

The result below shows that "✓✓" for $T = 2t-1$ implies "✓✓" for $T = 2t+1$.

Claim 9 *If seller 2 sells in period 1 of $G(2, B, T-1)$ and $G(B, 2, T)$ with $T = 2t-1 \geq 3$, then*

i) seller 2 sells in period 1 of $G(2, B, T+1)$ and $G(B, 2, T+2)$

ii) the mall is built for horizon $T = 2t+1$

iii) the buyer bargains with seller 2 first in the two-seller game

iv) (30) holds for $T = 2t+1$

Proof. We first derive several properties in Steps I-III, then use them to prove the claim in Step IV.

Step I. For any odd $T \geq 3$, if seller 2 in period 1 of $G(2, B, 2t-2)$ and $G(B, 2, 2t-1)$, we derive the condition under which seller 2 sells in period 1 of $G(2, B, 2t)$ and $G(B, 2, 2t+1)$. The analysis is similar to that deriving (13). Specifically, in period 1 of $G(2, B, 2t)$, the buyer is indifferent between accepting and rejecting $q^2_{2,2t}$ if $\delta\pi^1_{B,2t-1} - q^2_{2,2t} = \delta(\delta\pi^1_{B,2t-2} - p^2_{2,2t-1})$. In addition, seller 2 offers such a price if $q^2_{2,2t} - v_2 \geq \delta(p^2_{2,2t-1} - v_2)$, which means his surplus from the offer is no lower than that from waiting one period. The two conditions above imply that seller 2 sells in period 1 of $G(2, B, 2t)$ if and only if $\delta\pi^1_{B,2t-1} - \delta(\delta\pi^1_{B,2t-2} - p^2_{2,2t-1}) - v_2 \geq \delta(p^2_{2,2t-1} - v_2)$,

[4] Claim 1 is used to show $\pi^2_{B,3} > \pi^{2\prime}_{B,3}$ in Claim 7, so it is also needed to show $\pi^2_{B,2t+1} > \pi^{2\prime}_{B,2t+1}$.

or equivalently

$$\delta\pi^1_{B,2t-1} - v_2 \geq \delta(\delta\pi^1_{B,2t-2} - v_2) \tag{34}$$

This means that the total surplus of the buyer and seller 2 in $G(2, B, 2t)$ is no lower than the present value of their surplus in $G(B, 2, 2t-1)$.

In period 1 of $G(B, 2, 2t+1)$, seller 2 is indifferent between accepting and rejecting $p^2_{2,2t+1}$ if it satisfies $p^2_{2,2t+1} - v_2 = \delta(q^2_{2,2t} - v_2)$. The buyer offers such a price if $\delta\pi^1_{B,2t} - p^2_{2,2t+1} \geq \delta(\delta\pi^1_{B,2t-1} - q^2_{B,2t})$. These two conditions imply that seller 2 sells in period 1 of $G(2, B, 2t+1)$ if and only if $\pi^1_{B,2t} - v_2 - \delta(q^2_{2,2t} - v_2) \geq \delta(\delta\pi^1_{B,2t-1} - q^2_{B,2t})$, or equivalently

$$\delta\pi^1_{B,2t} - v_2 \geq \delta(\delta\pi^1_{B,2t-1} - v_2) \tag{35}$$

This means that the total surplus of the buyer and seller 2 in $G(B, 2, 2t+1)$ is no lower than the present value of their surplus in $G(2, B, 2t)$.

Next, we verify that (35) implies (34). Rewrite (35) as $(\delta\pi^1_{B,2t} - \delta^2\pi^1_{B,2t-1})/(1-\delta) \geq v_2$ and (34) as $(\delta\pi^1_{B,2t-1} - \delta^2\pi^1_{B,2t-2})/(1-\delta) \geq v_2$. Therefore, it is sufficient to show $\pi^1_{B,2t-1} - \delta\pi^1_{B,2t-2} \geq \pi^1_{B,2t} - \delta\pi^1_{B,2t-1}$. According to Claim 1, $\pi^1_{B,2t-1} > \pi^1_{B,2t}$ and $\pi^1_{B,2t-2} < \pi^1_{B,2t-1}$, so $\pi^1_{B,2t-1} - \delta\pi^1_{B,2t-2} \geq \pi^1_{B,2t} - \delta\pi^1_{B,2t-1}$. Therefore, seller 2 sells in period 1 of $G(2, B, 2t)$ and $G(B, 2, 2t+1)$ if (35) holds.

Step II. For any odd $T \geq 3$, if seller 2 sells in period 1 of $G(2, B, T-1)$ and $G(B, 2, T)$, and if the buyer bargains with seller 2 first when the horizon is $T+2$, seller 2 sells in period 1 of $G(2, B, T+1)$ and $G(B, 2, T+2)$. In Figure 2, this property means that "✓✓" for T implies "✓✓" for $T+2$.

We first prove the property for $T = 3$. According to Claim 7, if seller 2 sells in period 1 of $G(2, B, 2)$ and $G(B, 2, 3)$, (31) holds for $T = 3$, which is equivalent to

$$\delta\pi^1_{B,2} - \delta^2\pi^1_{B,1} \geq (1-\delta)v_2 \tag{36}$$

Step I implies that seller 2 sells in period 1 of $G(2, B, 4)$ and $G(B, 2, 5)$ if (35) holds for $t = 2$, i.e., $\delta\pi^1_{B,4} - v_2 \geq \delta(\delta\pi^1_{B,3} - v_2)$, or

$$\delta\pi^1_{B,4} - \delta^2\pi^1_{B,3} \geq (1-\delta)v_2 \tag{37}$$

According to Claim 1, $\delta\pi^1_{B,4} > \delta\pi^1_{B,2}$ and $\delta\pi^1_{B,3} < \delta\pi^1_{B,1}$, so (36) implies (37) with a strict inequality. This means the statement in Step II is true for $T = 3$.

Next we prove the property for $T = 5$. We first show that if seller 2 sells in period 1 of $G(2, B, 4)$ and $G(B, 2, 5)$, then (37) holds. The above analysis already proves the statement if seller 2 sells in period 1 of $G(2, B, 2)$ and $G(B, 2, 3)$. Now we prove it if seller 2 does not sell in period 1 of $G(2, B, 2)$ or $G(B, 2, 3)$. Then, Claim 7 implies the mall is not built if $T = 3$. In addition, Claim 8 implies that seller 2 sells in period 1 of $G(2, B, 4)$ and $G(B, 2, 5)$ if (31) holds for $T = 5$, i.e., $\delta\pi^1_{B,4} - v_2 \geq \delta(\delta\pi^1_{B,3} - v_2)$, which is also (37). Therefore, whether seller 2 sells in period 1 of $G(2,B,2)$ and $G(B,2,3)$, if seller 2 sells in period 1 of $G(2,B,4)$ and $G(B,2,5)$, (37) holds.

Similar to (37), Step I implies that seller 2 sells in period 1 of $G(2,B,6)$ and $G(B,2,7)$ if

$$\delta\pi^1_{B,6} - \delta^2\pi^1_{B,5} \geq (1-\delta)v_2 \tag{38}$$

As shown above, if seller 2 sells in period 1 of $G(2,B,4)$ and $G(B,2,5)$, we have (37). According to Claim 1, $\pi^1_{B,6} > \pi^1_{B,4}$ and $\pi^1_{B,5} < \pi^1_{B,3}$, so (37) implies (38) with a strict inequality. This means the property in Step II is true for $T=5$.

More generally, if seller 2 sells in period 1 of $G(2,B,T-1)$ and $G(B,2,T)$ for an odd $T \geq 3$, then $\delta\pi^1_{B,T-1} - \delta^2\pi^1_{B,T-2} \geq (1-\delta)v_2$, which implies a strict inequality $\delta\pi^1_{B,T+1} - \delta^2\pi^1_{B,T} > (1-\delta)v_2$. Then, Step I implies that seller 2 sells in period 1 of $G(2,B,T+1)$ and $G(B,2,T+2)$. Therefore, the property is true for any odd $T \geq 3$.

Step III. If seller 2 sells in period 1 of $G(2,B,2t-2)$ and $G(B,2,2t-1)$, the buyer prefers to bargain with seller 2 first if $T = 2t+1$. To prove this, we first derive the buyer's payoff if she purchases from seller 2 first, and then show that it exceeds her payoff if she purchases from seller 1 first.

First, if the buyer purchases from seller 2 first when $T = 2t+1$, her payoff is

$$\pi^2_{B,2t+1} = (\delta\pi^1_{B,2t} - v_2) - \delta(\delta\pi^1_{B,2t-1} - v_2) + \delta^2\pi^2_{B,2t-1} \tag{39}$$

Notice that if seller 2 sells in period 1 of $G(2,B,2t-2)$ and $G(B,2,2t-1)$, according to Step II, seller 2 sells in period 1 in $G(2,B,2t)$ and $G(B,2,2t+1)$. Then, in $G(2,B,2t)$, seller 2 offers price $q^2_{2,2t}$ such that the buyer is indifferent between accepting and rejecting, i.e., $\delta\pi^1_{B,2t-1} - q^2_{2,2t} = \delta\pi^2_{B,2t-1}$. In $G(B,2,2t+1)$, the buyer offers price $p^2_{2,2t+1}$ such that seller 2 is indifferent between accepting and rejecting, i.e., $p^2_{2,2t+1} - v_2 = \delta(q^2_{2,2t} - v_2)$. Because both offers are accepted, we can substitute $p^2_{2,2t+1}$ and $q^2_{2,2t}$ into $\pi^2_{B,2t+1} = \delta\pi^1_{B,2t} - p^2_{2,2t+1}$ and obtain (39).

Second, if the buyer purchases from seller 1 first, and if seller 1 sells in period 1 of $G(1,B,2t)$ and $G(B,1,2t+1)$, then the buyer's payoff is lower than that in (39). To see why, notice that in $G(1,B,2t)$, seller 1 offers a price $q^2_{1,2t}$ such that the buyer is indifferent between accepting and rejecting, i.e., $\delta\pi^{1\prime}_{B,2t-1} - q^2_{1,2t} = \delta\pi^2_{B,2t-1}$. In $G(B,1,2t+1)$, the buyer offers a price $p^2_{1,2t+1}$ such that seller 1 is indifferent between accepting and rejecting, i.e., $p^2_{1,2t+1} - v_1 = \delta(q^2_{1,2t} - v_1)$. Substituting $p^2_{1,2t+1}$ and $q^2_{1,2t}$ into $\pi^{2\prime}_{B,2t+1} = \delta\pi^{1\prime}_{B,2t} - p^2_{1,2t+1}$, we obtain

$$\pi^{2\prime}_{B,2t+1} = (\delta\pi^{1\prime}_{B,2t} - v_1) - \delta(\delta\pi^{1\prime}_{B,2t-1} - v_1) + \delta^2\pi^2_{B,2t-1} \tag{40}$$

Comparing (39) and (40), in order to show $\pi^{2\prime}_{B,2t+1} < \pi^2_{B,2t+1}$, we only need to verify

$$(\delta\pi^{1\prime}_{B,2t} - v_1) - \delta(\delta\pi^{1\prime}_{B,2t-1} - v_1) < (\delta\pi^1_{B,2t} - v_2) - \delta(\delta\pi^1_{B,2t-1} - v_2) \tag{41}$$

According to Claim 1, the LHS of this inequality can be rewritten as

$$\text{LHS of (41)} = \delta\alpha^1_{1,2t}(1-v_2) - v_1 - \delta(\delta\alpha^1_{1,2t-1}(1-v_2) - v_1)$$

Moreover, $\alpha^1_{1,2t} < \alpha^1_{1,2t-1} < 1$ according to Claim 1, so the coefficient of v_2 in the LHS has a smaller absolute value than that of v_1. As a result, the value is smaller if we switch v_1 and v_2 in LHS of (41). If we switch v_1 and v_2, the LHS of (41) becomes its RHS, so (41) is true.

Third, if the buyer purchases from seller 1 first, and if seller 1 sells in period 1 of $G(1, B, 2t)$ but not in period 1 of $G(B, 1, 2t+1)$, then the buyer's payoff is lower than that in (39). To see why, notice that in $G(1, B, 2t)$, seller 1 offers such that the buyer is indifferent between accepting and rejecting, which means the buyer's payoff is $\delta\pi^2_{B,2t-1}$. Therefore, the buyer's payoff in $G(B, 1, 2t+1)$ is $\delta^2\pi^2_{B,2t-1}$. Notice that $\delta^2\pi^2_{B,2t-1}$ is the third term in (39), so in order to show $\delta^2\pi^2_{B,2t-1} < \pi^2_{B,2t+1}$, it is sufficient to verify the sum of the first two terms in (39) is positive, i.e., $(\delta\pi^1_{B,2t} - v_2) - \delta(\delta\pi^1_{B,2t-1} - v_2) > 0$. This inequality is equivalent to $\delta\pi^1_{B,2t} - \delta^2\pi^1_{B,2t-1} > (1-\delta)v_2$. If seller 2 sells in period 1 of $G(1, B, 2t-2)$ and $G(B, 1, 2t-1)$, which is the qualifier in the statement of Step III, we show in Step II above that $\delta\pi^1_{B,2t} - \delta^2\pi^1_{B,2t-1} > (1-\delta)v_2$. Therefore, the buyer's payoff $\delta^2\pi^2_{B,2t-1}$ is lower than that in (39).

Fourth, if the buyer purchases from seller 1 first, and if seller 1 does not sell in period 1 of $G(1, B, 2t)$ but sells in period 1 of $G(B, 1, 2t+1)$, the buyer's payoff is lower than that in (39). In $G(1, B, 2t)$, the buyer is indifferent between accepting and rejecting $q^2_{1,2t}$ if

$$\delta\pi^{1\prime}_{B,2t-1} - q^2_{1,2t} = \delta\pi^2_{B,2t-1} \tag{42}$$

Notice that there is no agreement in period 1 of $G(1, B, 2t)$, so seller 1 does not offer such a price, which implies $q^2_{1,2t} - v_1 < \delta^2(p^2_{1,2t-2} - v_1)$. Solving $q^2_{1,2t}$ from (42) and substituting into the above inequality, we can rewrite the inequality as

$$\delta\pi^{1\prime}_{B,2t-1} - v_1 - \delta\pi^2_{B,2t-1} < \delta^2(p^2_{1,2t-2} - v_1) \tag{43}$$

In $G(B, 1, 2t+1)$, the buyer offers $p^2_{1,2t+1}$ such that seller 1 is indifferent between accepting and rejecting, i.e., $p^2_{1,2t+1} - v_1 = \delta^3(p^1_{1,2t-2} - v_1)$. Solving $p^2_{1,2t+1}$ from the this equation and substituting it into the buyer's payoff in $G(B, 1, 2t+1)$, we obtain

$$\begin{aligned}
\pi^{2\prime}_{B,2t+1} &= \delta\pi^{1\prime}_{B,2t} - p^2_{1,2t+1} \\
&= (\delta\pi^{1\prime}_{B,2t} - v_1) - \delta^3(p^1_{1,2t-2} - v_1) \\
&< (\delta\pi^{1\prime}_{B,2t} - v_1) - \delta(\delta\pi^{1\prime}_{B,2t-1} - v_1) + \delta^2\pi^2_{B,2t-1} \\
&< (\delta\pi^1_{B,2t} - v_2) - \delta(\delta\pi^1_{B,2t-1} - v_2) + \delta^2\pi^2_{B,2t-1} \\
&= \pi^2_{B,2t+1}
\end{aligned}$$

where the first inequality is from (43), the second from (41) and the last equality from (39).

So far we discuss all the cases in which the buyer purchases from seller 1 first, and show that the buyer's payoff in each case is lower than that in (39). Hence, the statement in Step III is true.

Step IV. We prove Claim 9 by induction. If seller 2 sells in period 1 of $G(2, B, 2t-2)$ and $G(B, 2, 2t-1)$ with $2t-1 \geq 3$, Step II implies that if the buyer chooses to bargain with seller 2 first when horizon is $T+2$, seller 2 sells in period 1 of $G(2, B, 2t)$ and $G(B, 2, 2t+1)$. Moreover, Step III implies that in the subgame given participation the buyer indeed chooses seller 2 to bargain with first. In the third point in Step III, we verify that the buyer's payoff in (39) is indeed positive by purchasing from seller 2 first when $T = 2t+1$. Thus, the mall is built for $T = 2t+1$ if seller 2 sells in period 1 of $G(2, B, 2t-2)$ and $G(B, 2, 2t-1)$ with $T = 2t-1$. Finally, also in the third point in Step III, we verify that $\delta\pi^1_{B,2t} - \delta^2\pi^1_{B,2t-1} > (1-\delta)v_2$, which is (30) for $T = 2t+1$. ■

Proof of Proposition 1 for $N = 2$ and $T = 2t+1$.

Consider a two-seller game with sellers 1 and 2. We start with $T = 3$, then Claim 7 implies Proposition 1. In addition, Claim 7 implies that if $T = 3$, the mall is built if and only if (30) holds. First, consider the case in which the mall is not built when $T = 3$. Then, Claim 8 implies that when $T = 5$, the mall is built if and only if (30) holds for $T = 5$. Moreover, if the mall is built, the buyer purchases from seller 2 in period 1. Therefore, whether the mall is built, the two-seller game has the unique equilibrium outcome described as in Proposition 1, so the proposition holds for $T = 5$.

Second, consider the other case in which the mall is built when $T = 3$. Then, Claim 7 implies seller 2 sells in period 1 of $G(2, B, T-1)$ and $G(B, 2, T)$ for $T = 3$. Then, Claim 9 implies the mall is built when $T = 5$, the buyer bargains with seller 2 first, and seller 2 sells in period 1 of $G(2, B, T-1)$ and $G(B, 2, T)$. Hence, the two-seller game has a unique equilibrium outcome described as in Proposition 1, so the proposition holds for $T = 5$.

Using the property that the mall is built if and only if (30) holds for $T = 3$, we proves the proposition for $T = 5$. Moreover, Claims 8 and 9 imply that the mall is built if and only if (30) holds for $T = 5$. Hence, more generally, using the property that the mall is built if and only if (30) holds for $T = 2t+1 \geq 3$, we can show the proposition for $T = 2t+3$ and that the mall is built if and only if (30) holds for $T = 2t+3$. Therefore, Proposition 1 holds in the two-seller game with any odd horizon. ■

To complete the analysis for odd horizons, the following result summarizes the evolution of different cases and the condition for the mall to be built.

Claim 10 *In the two-seller game given participation with horizon $T = 2t+2 \geq 4$,*

i) case "✓✓" arises if (31) holds

ii) case "×✓" or "××" arises if (31) does not hold

iii) "✓✓" for $T-2$ implies "✓✓" for T

iv) the mall is built if and only if (31) holds with a strict inequality

Proof. According to Claims 7 and 8, case "✓✓" arises if (31) holds, and case "×✓" or "××" arises otherwise. Thus, i) and ii) hold. Moreover, Claim 9 implies that iii) holds. We show, in Claim 7, that the mall is built if and only if (30) holds for $T = 3$, and in Claims 8 and 9 that, for $T = 2t + 1 > 3$, the mall is built if and only if (30) holds, which is exactly (31) with a strict inequality. xHence, iv) is also true. ■

2 The Proof of Proposition 1 for $N > 2$

Using the properties of the one-seller game, we prove Proposition 1 in the two-seller game. All these properties are summarized in Claim 1. Next, we generalize these properties to the two-seller game in the claim below, and then use them to prove Proposition 1 in the three-seller game.

Claim 11 *In the two-seller game with sellers 1 and 2, the mall is not built for horizon $T = 2$. If the mall is built for horizon $T \geq 3$,*

i) the buyer's equilibrium payoff $\pi^2_{B,T}$ is a linear function of v_1 and v_2

ii) $\alpha^2_{2,T+2} = 1 - \delta + \delta^2 \alpha^2_{2,T}$ and $0 < \alpha^2_{1,T} < \alpha^2_{2,T} < 1$, where $\alpha^2_{i,T}$ is the absolute value of v_i's coefficient in $\pi^2_{B,T}$

iii) $\pi^2_{B,T+2} = (\delta\pi^1_{B,T+1} - v_2) - \delta(\delta\pi^1_{B,T} - v_2) + \delta^2\pi^2_{B,T}$

iv) $\pi^2_{B,T} > \pi^2_{B,T+2}$ if T is even, and $\pi^2_{B,T} < \pi^2_{B,T+2}$ if T is odd

Proof. Consider even horizons first. Suppose the mall is built for sellers with v_1 and v_2 and T, then let T^2_e be the longest even horizon such that the mall is not built for these sellers. Next, we show that in the buyer's payoff is

$$\pi^2_{B,T^2_e} = \delta\pi^1_{B,T^2_e-1} - v_2 \tag{44}$$

in the subgame given participation with horizon T^2_e. To see this, notice that for $T^2_e = 2$, Claim 2 implies $\pi^2_{B,2} = \delta\pi^1_{B,2} - v_2$, so (44) is true. If $T^2_e > 2$, case "✓×" arises for horizons $4, 6, ..., T^2_e$ according to Claim 6. In the proof of Claim 6, we show that seller 2's surplus is zero for these horizons, so the buyer's payoff is the total surplus left for her and seller 2, which is $\pi^2_{B,T^2_e} = \delta\pi^1_{B,T^2_e-1} - v_2$.

Claim 6 implies that for any even horizon longer than T^2_e, the mall is built. Next, we prove the properties in Claim 11 for horizon $T^2_e + 2$. For $T = T^2_e + 2$, case "✓✓" arises and Claim 4 implies that

$$\pi^2_{B,T^2_e+2} = \delta\pi^1_{B,T^2_e+1} - v_2 - \delta(\delta\pi^1_{B,T^2_e} - v_2) + \delta^2\pi^2_{B,T^2_e} \tag{45}$$

which means iii) holds. Claim 1 implies that both π^1_{B,T^2_e+1} and π^1_{B,T^2_e} are linear function of v_1, which combined with (44) implies π^2_{B,T^2_e+2} is a linear function of v_1 and v_2, so i) also holds. Because the mall is built for horizon $T^2_e + 2$, so every player's surplus is positive. Then, $0 < \alpha^2_{1,T^2_e+2}$, otherwise seller 1's surplus is not positive. In addition, $\alpha^2_{2,T^2_e+2} < 1$, otherwise the buyer's surplus is zero. As a result, to show ii) for horizon T_e^2+2, it remains to verify $\alpha^2_{1,T_e^2+2} < \alpha^2_{2,T_e^2+2}$. In (45), we have $\alpha^2_{2,T_e^2+2} = 1-\delta+\delta^2\alpha^2_{2,T_e^2}$ and $\alpha^2_{1,T_e^2+2} = \delta\alpha^1_{1,T_e^2+1} - \delta^2\alpha^1_{1,T_e^2} + \delta^2\alpha^2_{1,T_e^2}$. From the expression of π^2_{B,T_e^2}, we have $\alpha^2_{1,T_e^2} < \alpha^2_{2,T_e^2}$. Therefore, for ii) to hold for $T = T_e^2+2$, it is sufficient to show

$$\delta\alpha^1_{1,T_e^2+1} - \delta^2\alpha^1_{1,T_e^2} < 1-\delta \tag{46}$$

Because the mall is built for horizon T_e^2+2, condition (29) in Claim 6 implies $\delta\pi^1_{B,T_e^2} - v_2 \geq \delta(\delta\pi^1_{B,T_e^2-1} - v_2)$. Moving v_2 to one side, this inequality becomes $\delta\pi^1_{B,T_e^2} - \delta^2\pi^1_{B,T_e^2-1} \geq (1-\delta)v_2 > 0$, so

$$\begin{aligned}\delta < \frac{\pi^1_{B,T_e^2}}{\pi^1_{B,T_e^2-1}} &= \frac{(1-v_1)(1-\delta+...+(-1)^{T_e^2-1}\delta^{T_e^2-1})}{(1-v_1)(1-\delta+...+(-1)^{T_e^2-2}\delta^{T_e^2-2})} \\ &= \frac{1-\delta+...+(-1)^{T_e^2-1}\delta^{T_e^2-1}}{1-\delta+...+(-1)^{T_e^2-2}\delta^{T_e^2-2}}\end{aligned} \tag{47}$$

where the first equality is from Claim 1. Moreover, Claim 1 also implies

$$\begin{aligned}\delta\alpha^1_{1,T_e^2+1} &= \delta[1-\delta+...+(-1)^{T_e^2-1}\delta^{T_e^2}] \\ &= \delta(1-\delta)+\delta^2[1-\delta+...+(-1)^{T_e^2-1}\delta^{T_e^2-2}] \\ &< 1-\delta+\delta^2[1-\delta+...+(-1)^{T_e^2-1}\delta^{T_e^2-1}] \\ &= 1-\delta+\delta^2\alpha^1_{1,T_e^2}\end{aligned}$$

where the inequality is from (47). Hence, (46) holds and ii) holds for $T = T_e^2+2$.

Next, we prove iv) for $T = T_e^2+2$. Substituting (44) into (45), we obtain

$$\begin{aligned}\pi^2_{B,T_e^2+2} &= (\delta\pi^1_{B,T_e^2+1} - v_2) - \delta(\delta\pi^1_{B,T_e^2} - v_2) + \delta^2(\delta\pi^1_{B,T_e^2-1} - v_2) \\ &= (\delta\pi^1_{B,\infty} - v_2)(1-\delta+\delta^2) \\ &\quad +\delta(\pi^1_{B,T_e^2+1} - \pi^1_{B,\infty}) - \delta^2(\pi^1_{B,T_e^2} - \pi^1_{B,\infty}) + \delta^3(\pi^1_{B,T_e^2-1} - \pi^1_{B,\infty}) \\ &= (\delta\pi^1_{B,\infty} - v_2)(1-\delta+\delta^2) + \sum_{t=1}^{3}\underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T_e^2+2-t} - \pi^1_{B,\infty})]}_{>0}\end{aligned}$$

where the last term is positive because of iv) of Claim 1. Similar to (45), we have

$$\begin{aligned}\pi^2_{B,T_e^2+4} &= \delta\pi^1_{B,T_e^2+3} - v_2 - \delta(\delta\pi^1_{B,T_e^2+2} - v_2) + \delta^2\pi^2_{B,T_e^2+2} \\ &= (\delta\pi^1_{B,\infty} - v_2)(1-\delta+\delta^2-\delta^3+\delta^4) + \sum_{t=1}^{5}\underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T_e^2+4-t} - \pi^1_{B,\infty})]}_{>0}\end{aligned}$$

Therefore,

$$\begin{aligned}\pi^2_{B,T^2_e+2} - \pi^2_{B,T^2_e+4} &= (\delta\pi^1_{B,\infty} - v_2)(\delta^3 - \delta^4) + \sum_{t=1}^{3}[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+2-t} - \pi^1_{B,\infty})] \\ &\quad - \sum_{t=1}^{5}[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+4-t} - \pi^1_{B,\infty})] \end{aligned} \tag{48}$$

Recall that we obtain $\delta\pi^1_{B,T^2_e} - v_2 \geq \delta(\delta\pi^1_{B,T^2_e-1} - v_2)$ below (47), so

$$(\delta\pi^1_{B,\infty} - v_2)(1-\delta) \geq -\delta(\pi^1_{B,T^2_e} - \pi^1_{B,\infty}) + \delta^2(\pi^1_{B,T^2_e-1} - \pi^1_{B,\infty})$$

Substituting this inequality into (48), we obtain

$$\begin{aligned} & \pi^2_{B,T^2_e+2} - \pi^2_{B,T^2_e+4} \\ \geq\ & \underbrace{-\delta^4(\pi^1_{B,T^2_e} - \pi^1_{B,\infty})}_{>0} + \underbrace{\delta^5(\pi^1_{B,T^2_e-1} - \pi^1_{B,\infty})}_{>0} + \sum_{t=1}^{3}[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+2-t} - \pi^1_{B,\infty})] \\ & +\delta^4\underbrace{(\pi^1_{B,T^2_e} - \pi^1_{B,\infty})}_{<0} - \delta^5\underbrace{(\pi^1_{B,T^2_e-1} - \pi^1_{B,\infty})}_{>0} - \sum_{t=1}^{3}[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+4-t} - \pi^1_{B,\infty})] \\ =\ & \sum_{t=1}^{3}\underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+2-t} - \pi^1_{B,\infty})]}_{>0} - \sum_{t=1}^{3}\underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_e+4-t} - \pi^1_{B,\infty})]}_{>0} \\ >\ & 0 \end{aligned}$$

where the last inequality is from iv) of Claim 1. Therefore, iv) is also true for $T = T^2_e + 2$. We have shown properties i) to iv) for $T = T^2_e + 2$. Using the same argument, we can prove these properties for even horizons longer than $T^2_e + 2$.

Consider odd horizons. Suppose the mall is built for sellers with v_1 and v_2 and an odd horizon T, then let $T^2_o \geq 3$ be the longest odd horizon such that the mall is not built for these sellers. Next, we show that in the buyer's payoff is $\pi^2_{B,T^2_o} = 0$. To see this, recall that in the proof of Claim 7, we show that the buyer's payoff is zero for $T = 3$ in the cases "××" and "×✓". Then, if the horizon is $T = 5$, the buyer's payoff is zero in case "××". Moreover, if case "×✓" arises for $T = 5$, seller 2 offers such that the buyer is indifferent between accepting and rejecting, so the buyer's payoff for $T = 5$ is $\delta^2\pi^2_{B,3}$. Claim 8 implies that if one of the two cases arises for $T = 5$, then one of the two cases arises for $T = 3$. In either case, $\pi^2_{B,3} = 0$, so if case "×✓" arises for $T = 5$, the buyer's payoff is zero. Hence, the buyer's payoff is zero for $T = 5$ in the cases "××" and "×✓". Similarly, the buyer's payoff is zero for horizon T^2_o because one of the two cases arises. That is, $\pi^2_{B,T^2_o} = 0$.

Then, by the same argument deriving (39), we have

$$\begin{aligned}\pi^2_{B,T^2_o+2} &= \delta\pi^1_{B,T^2_o+1} - v_2 - \delta(\delta\pi^1_{B,T^2_o} - v_2) + \delta^2\pi^2_{B,T^2_o} \\ &= \delta\pi^1_{B,T^2_o+1} - v_2 - \delta(\delta\pi^1_{B,T^2_o} - v_2) \end{aligned} \tag{49}$$

Recall that both π^1_{B,T^2_o+1} and π^1_{B,T^2_o} are linear functions of v_1, so π^2_{B,T^2_o+2} above is a linear function of v_1 and v_2, which proves i) for $T = T^2_o + 2$. Moreover, according to Claim 1, the absolute values of the coefficients of v_1 in π^1_{B,T^2_o+1} and π^1_{B,T^2_o} are α^1_{1,T^2_o+1} and α^1_{1,T^2_o} respectively. Then, the absolute value of v_1 in (49) is $\alpha^2_{1,T^2_o+2} = \delta\alpha^1_{1,T^2_o+1} - \delta^2\alpha^1_{1,T^2_o} < \delta\alpha^1_{1,T^2_o}(1-\delta) < 1-\delta = \alpha^2_{2,T^2_o+2}$, where the inequalities are from $\alpha^1_{1,T^2_o+1} < \alpha^1_{1,T^2_o} < 1$ for the odd horizon T^2_o according to Claim 1. In addition, $\alpha^2_{1,T^2_o+2} > 0$, otherwise seller 1 receives a zero surplus and does not participate, and $\alpha^2_{1,T^2_o} = 1-\delta < 1$. Hence, $0 < \alpha^2_{1,T^2_o+2} < \alpha^2_{2,T^2_o+2} < 1$. By the same argument derving (39), we have

$$\pi^2_{B,T^2_o+4} = \delta\pi^1_{B,T^2_o+3} - v_2 - \delta(\delta\pi^1_{B,T^2_o+2} - v_2) + \delta^2\pi^2_{B,T^2_o+2} \tag{50}$$

so $\alpha^2_{2,T^2_o+4} = 1 - \delta + \delta^2\alpha^2_{2,T^2_o+2}$. Hence, ii) holds for $T = T^2_o + 2$. Equation (50) also implies iii) holds for $T = T^2_o + 2$. Subsituting (49) into (50), we obtain

$$\begin{aligned}\pi^2_{B,T^2_o+4} &= \delta\pi^1_{B,T^2_o+3} - v_2 - \delta(\delta\pi^1_{B,T^2_o+2} - v_2) + \delta^2(\delta\pi^1_{B,T^2_o+1} - v_2) - \delta^3(\delta\pi^1_{B,T^2_o} - v_2) \\ &= (\delta\pi^1_{B,\infty} - v_2)(1 - \delta + \delta^2 - \delta^3) + \sum_{t=1}^{4} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+4-t} - \pi^1_{B,\infty})]}_{<0}\end{aligned}$$

Similarly, we can rewrite (49) as

$$\pi^2_{B,T^2_o+2} = (\delta\pi^1_{B,\infty} - v_2)(1-\delta) + \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} \tag{51}$$

Therefore,

$$\begin{aligned}\pi^2_{B,T^2_o+4} - \pi^2_{B,T^2_o+2} &= (\delta\pi^1_{B,\infty} - v_2)(\delta^2 - \delta^3) + \sum_{t=1}^{4} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+4-t} - \pi^1_{B,\infty})]}_{<0} \\ &\quad - \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0}\end{aligned} \tag{52}$$

Recall that the mall is built for horizon $T^2_o + 2$, so the buyer's payoff must be positive, which combined with (51) implies

$$(\delta\pi^1_{B,\infty} - v_2)(1-\delta) \;>\; -\sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0}$$

where the second inequality is from iv) in Claim 1 for odd horizons. Substituting this inequality into (52), we have

$$
\begin{aligned}
& \pi^2_{B,T^2_o+4} - \pi^2_{B,T^2_o+2} \\
> & \sum_{t=1}^{4} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+4-t} - \pi^1_{B,\infty})]}_{<0} \\
& - \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} - \delta^2 \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} \\
= & \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+4-t} - \pi^1_{B,\infty})]}_{<0} + \delta^2 \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} \\
& - \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} - \delta^2 \sum_{t=1}^{2} \underbrace{[(-1)^{t-1}\delta^t(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})]}_{<0} \\
> & \; 0
\end{aligned}
$$

where the last inequality is from $0 > (-1)^{t-1}(\pi^1_{B,T^2_o+4-t} - \pi^1_{B,\infty}) > (-1)^{t-1}(\pi^1_{B,T^2_o+2-t} - \pi^1_{B,\infty})$ due to Claim 1. Therefore, iv) is also true for $T = T^2_o + 2$. We have shown properties i) to iv) for $T = T^2_o + 2$. Using the same argument, we can prove these properties for even horizons longer than $T^2_o + 2$. ■

Remark 1 *Note that the orders of $\pi^1_{B,T}$ and $\pi^2_{B,T}$ are different. In the one-seller game, $\pi^1_{B,T}$ for an odd T is larger than $\pi^1_{B,T'}$ for an even T'. In the two-seller game, $\pi^2_{B,T}$ for an odd T is lower than $\pi^1_{B,T'}$ for an even T'. To see why, notice that if $T = N$, there is exactly one period for each seller, so there is no time for the sellers to counter offers. As a result, the buyer's bargaining power is the largest for $T = N$, as the power reduces as the horizon becomes longer.*

Proof of Proposition 1 for $N \geq 3$. The proofs of Claims 2-9 are readily to be extended to the N-seller games except that instead of the properties in Claim 1, we need to use their counterparts in Claim 11. Recall that we use the properties in the one-seller game (in Claim 1) to prove the proposition in the two-seller game, so we need to use the properties in the two-seller game (in Claim 11) to show the proposition in the three-seller game. As a result, we only sketch the proof for $N = 3$ below.

For odd horizons $T \geq 3$, we also have three cases "$\times\times$", "$\checkmark\times$" and "$\checkmark\checkmark$" as in Figure 1. More generally, for even horizons for an even number of sellers and for odd horizons for an odd number of sellers, there three cases as in Figure 1. Note that in the three-player game, these cases refer to agreement between the buyer and the smallest seller 3. Then, following the same analysis in Claims 2-5, we can prove Proposition 1 for three sellers and odd horizons.

For even horizons $T \geq 4$, we have three cases "$\times\times$", "$\times\checkmark$" and "$\checkmark\checkmark$" as in Figure 2. Then, following the same analysis in Claims 7-9, we can prove Proposition 1 for three sellers and even horizons.

So far, we use the properties in the one-seller (two-seller) game to prove Proposition 1 in the two-seller (three-seller) game. Next, we generalize the properties in Claim 11 to the three-seller game. For $N \geq 3$, the generalized Claim 11 is:

In the N-seller game with sellers $1, ..., N$, the mall is not built for horizon $T = N$. If the mall is built for horizon $T \geq N+1$,

i) the buyer's equilibrium payoff $\pi^N_{B,T}$ is a linear function of $v_1, ..., v_N$

ii) $\alpha^N_{N,T+2} = 1 - \delta + \delta^2 \alpha^N_{N,T}$ and $0 < \alpha^N_{N-1,T} < \alpha^N_{N,T} < 1$, where $\alpha^N_{i,T}$ is the absolute value of v_i's coefficient in $\pi^N_{B,T}$

iii) $\pi^N_{B,T+2} = (\delta\pi^{N-1}_{B,T+1} - v_N) - \delta(\delta\pi^{N-1}_{B,T} - v_N) + \delta^2\pi^N_{B,T}$

iv) $\pi^N_{B,T} > \pi^N_{B,T+2}$ if $T - N$ is even, and $\pi^N_{B,T} < \pi^N_{B,T+2}$ if $T - N$ is odd

The proof of Claim 11 generalizes to the three-seller game with one modification, which we describe below. To generalize the claim, we need to use properties in the two-seller game. However, unlike in the one-seller game, the mall may not be built for two or more sellers unless the horizon is long enough. See the definitions of T^2_e and T^2_o in the proof of Claim 11. Because of this difference, we need to adjust (46). The counterpart of (46) for $N = 3$ is $\delta < \pi^2_{B,T^3_o}/\pi^2_{B,T^3_o-1}$, where T^N_o is the longest odd horizon such that the mall is not built for selllers with v_1, v_2, v_3. Then,

$$\begin{aligned}
\delta &< \frac{\pi^2_{B,T^3_o}}{\pi^2_{B,T^3_o-1}} = \frac{\sum_{t=1}^{T^3_o-T^2_o}[(-1)^{t-1}\delta^{t-1}(\delta\pi^1_{B,T^3_o-t} - v_2)]}{\sum_{t=1}^{T^3_o-1-T^2_e}[(-1)^{t-1}\delta^{t-1}(\delta\pi^1_{B,T^3_o-1-t} - v_2)]} \\
&< \frac{(\delta\pi^1_{B,\infty} - v_2)(1 - \delta + ... + (-\delta)^{T^3_o-T^2_o-1})}{(\delta\pi^1_{B,\infty} - v_2)(1 - \delta + ... + (-\delta)^{T^3_o-T^2_e-2})} \\
&= \frac{1 - \delta + ... + (-\delta)^{T^3_o-T^2_o-1}}{1 - \delta + ... + (-\delta)^{T^3_o-T^2_e-2}} \qquad (53)
\end{aligned}$$

where the first equality is from iii) of Claim 11, and the second inequality is because, according to iv) in Claim 11, each term in the numerator $(-1)^{t-1}(\delta\pi^1_{B,T^3_o-t} - v_2) < \delta\pi^1_{B,\infty} - v_2$ for each t and each term in the denominator $(-1)^{t-1}(\delta\pi^1_{B,T^3_o-1-t} - v_2) > \delta\pi^1_{B,\infty} - v_2$. Then, as in the proof of Claim 11, we can use (53) to show $\alpha^3_{2,T} < \alpha^3_{3,T}$ for odd horizons, where $\alpha^N_{i,T}$ is the absolute value of v_i's coefficient in $\pi^N_{B,T}$.

So far, for $N = 1$ and 2, we use the properties in the N-seller game to prove Proposition 1 in the $(N+1)$-seller game, and generalize the properties to the $(N+1)$-seller game. Repeating the above analysis, for any $N \geq 3$, we can use the properties in the generalized Claim 11 for N sellers to prove Proposition 1 and the generalize Claim 11 in the $(N+1)$-seller game. Thus, Proposition 1 is also true for any $N \geq 3$. ■