Scarcity of Ideas and Optimal Prizes in Innovation Contests*

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August 16, 2021

Abstract

This paper studies the relationship between optimal prizes and scarcity of ideas in innovation contests. We consider a model where both ideas and effort are integral parts of the innovation process. Contest participants are privately informed about their idea quality. We introduce a new stochastic order to rank scarcity of ideas and study how a contest designer's choice–the profit maximizing prize–should vary with scarcity of ideas. We find that scarcity of ideas results in higher optimal prizes if and only if the benefit from a marginal improvement in the new technology's performance is sufficiently low.

JEL classification: D44, D82, O31

Keywords: innovation, idea distribution, contests, comparative statics, stochastic dominance

*We thank Suren Basov, Francis Bloch, Isa Hafalir, Vijay Krishna, Igor Letina, Priscilla Man, Claudio Mezzetti, Benny Moldovanu, Sergio Parreiras, Leo Simon, Satoru Takahashi, participants in 26th International Conference on Game Theory, 11th World Congress of Econometric Society, 18th Annual Society for the Advancement of Economic Theory Conference, and seminar participants at Deakin University, National University of Australia, National University of Singapore, Monash University, Penn State University, Queensland University of Technology, University of New South Wales, University of Melbourne, and University of Queensland for their comments and discussion. Jingong Huang and Eado Varon have provided excellent research assistance. Financial support from the Australian Research Council Discovery grant DP170103374 is greatly appreciated.

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1 Introduction

The process of innovation starts with a problem. What turns a problem into a world-changing discovery is an insight or an idea. Hence, an idea represents an investment opportunity and can be turned into an innovation by exerting effort.

Contests have a long history in the procurement of innovation. In the case of an innovation contest, the problem to be solved is defined by the contest designer and is public information. Historically, innovation contests were predominantly used by governments seeking solutions to major innovation problems faced by their countries.[1] More recently, designing contests to solve innovation problems has been growing in popularity in the private sector also. Even major industry players, such as BMW (Füller et al., 2006), Cisco, IBM (Bjelland and Wood, 2008), Philips, and Qualcomm, utilize contests to enhance their innovative capacity.[2]

In this paper, we study the design of optimal prizes in innovation contests. There exists a large variation in the prizes awarded in innovation contests. For example, the average prize awarded by XPRIZE, an organization running public innovation contests, was \$4.5 million in 1996-2014 and \$8.4 million in 2014-2018. As another example, Netflix awarded \$1 million in 2009 to the computer algorithm that best improves its recommendation system. In 2013, the company awarded a much lower prize of \$10,000 to those who most improve its cloud computing services. One explanation for this variation may be that some problems are more difficult than others, and therefore higher prizes are awarded to solve them. However, is it always optimal to incentivize more difficult challenges with higher prizes?

Our starting point is a model of innovation where the key ingredients are ideas and effort. To solve a problem, a solver (i.e., potential innovator) must first have an idea and then the incentive to invest in that idea. We assume that ideas can be ranked in terms of their quality. Different solvers privately receive ideas with different qualities for the same problem. They then decide whether to invest and how much to invest in solving the problem. The best solution wins the contest and receives the prize provided it is above a performance requirement.

Different problems in our model are represented by different idea distributions. Some problems are easy and then many innovators are likely to have high-quality ideas to start

[1]For example, the British Parliament offered a prize of £20,000 in 1714 for a method of determining longitude at sea. In 1795, the French military offered a cash prize of 12,000 francs for a new method to preserve food (which resulted in the development of canning).

[2]Innovation contests are also sometimes used by venture capitalists in the allocation of funds. See for example QPrize, which is Qualcomm Ventures' seed investment competition.

with. In this case, competition is more likely to take place along the effort dimension. As the problem gets more difficult, the probability of having a high-quality idea decreases. In this case, solvers are more likely to compete along the idea dimension, i.e., having a good idea will give a solver a strong advantage.

Scarcity is a fundamental concept in economics and is considered to be one of the main determinants of an object's value. Does this mean that the winner in a contest where high-quality ideas are scarcer should receive a higher prize? In other words, to what extent should we expect the prize to reflect the scarcity of high-quality ideas?[3] We show that the answer to this seemingly intuitive question is not straightforward.

To explore how the optimal prize changes as the scarcity of high-quality ideas changes, we introduce a new order of stochastic dominance to capture the notion of scarcity. An innovator has a scarcer idea if the probability that someone else has a better idea than him is smaller. Since this probability depends on the size of the solver base, the scarcity order also depends on the solver population size.

Our results uncover that the relationship between scarcity of high-quality ideas and the optimal prize critically depends on the market value of the innovation, specifically on the marginal value of solution performance for the seeker (i.e., contest designer). Consider two examples. The Wolfskehl Prize awards DM 100,000 to the first person to rediscover the proof of Fermat's Last Theorem. As long as a proof is correct, it serves the purpose of validating the theorem. Hence, the benefit of a proof depends only on whether it meets a minimal requirement. Compare this with the Netflix Prize, which awarded $1 million in 2009 to the best algorithm to predict user ratings for films. In this case, the minimum requirement was for the new algorithm to outperform the existing one. Otherwise, there was no benefit to Netflix. Beyond meeting this minimum requirement, the more an algorithm improved the existing one, the more valuable it was. The winning algorithm bested Netflix's own algorithm for predicting ratings by 10.06%.[4]

We show that if the market value is not very sensitive to the quality of the solution (i.e.,

[3]The prize in an innovation contest is similar to a patent. Through the patentability requirements, society selects which innovations should receive a patent (i.e., prize). For example, in US patent law, an innovation should satisfy the nonobviousness requirement. According to this requirement, an invention is considered nonobvious if someone with ordinary skill or training in the relevant field could not easily make the invention based on prior art. In European patent law, the same idea is captured by the inventive step requirement, which states that an invention should be sufficiently inventive in order to be patented. In both cases, an invention is considered to be patentable if it meets a legally defined "scarcity" requirement.

[4]The winning team BellKor's Pragmatic Chaos improved the predictions by 10.06% on the test data set, which Netflix used to determine the final winner. The contest structure for the Netflix Prize was different from the one we consider in this paper. It was dynamic, where the first submission triggered a deadline for further submissions. Nevertheless, the payoff function of the seeker is the same as ours.

as long as the solution meets the minimal quality, additional quality does not have much impact on the market value of the innovation), then the goal of the seeker is to maximize the probability of having a solution that meets the minimal quality requirement. In this case, as the scarcity of high-quality ideas increases, the seeker compensates by increasing the prize level. Hence, the optimal prize is increasing in the scarcity of high-quality ideas. However, if the marginal value of solution performance for the seeker is sufficiently high, then the goal of the seeker is to obtain as good a solution as possible. In this case, the optimal prize is decreasing in the scarcity of high-quality ideas. Intuitively, when the seeker cares highly about the solution performance, the expected return from a pool of solvers decreases as the scarcity of high-quality ideas increases. The seeker is willing to invest more in a contest where the likelihood of a high-performance solution is higher.

We consider three extensions of our model. The results generalize in a straightforward way to the case where the seeker has a nonlinear benefit function. As a second extension, we consider a variation of the model where the seeker sets the minimum performance requirement in addition to the prize. Finally, we generalize our model to allow for multiple prizes.

In general, our results imply that contest designers, while adjusting prize levels with the difficulty of challenges, should pay attention to how much they will benefit from a marginal increase in performance. Although our paper is couched in the language of innovation contests, our analysis applies to a wider range of scenarios. Our results have implications for any contest environment with private types where the participants have to exert effort and the contest designer cares about the best performance only. In other contexts, the reason for the private information may be past experience, access to different resources, genetic make-up, etc.

The remainder of the paper is organized as follows. Section 2 discusses the related literature. Section 3 introduces a model of contests and Section 4 characterizes the equilibrium strategies and the optimal prize. Section 5 presents our main result on how the optimal prize changes with the scarcity of ideas. Section 6 explores extensions of our benchmark model and shows the robustness of our main result. Finally, Section 7 concludes. All proofs are relegated to the appendix.

2 Related Literature

Our paper contributes to a large literature on contests with incomplete information. See, e.g., Moldovanu and Sela (2001), Chawla et al. (2015), Liu et al. (2018) and Olszewski and

Siegel (2020). There also exists a growing literature specifically on innovation contests. See, e.g., Taylor (1995), Fullerton and McAfee (1999), Fullerton et al. (2002), Che and Gale (2003), Terwiesch and Xu (2008) and Korpeoglu and Cho (2018).[5]

Our paper has a different focus from all these papers because we are interested in the question of how the optimal prize changes as the distribution of ideas changes. An important feature of our paper is that each participant privately has access to a solution idea that they can invest in. This allows us to focus on the importance of ideas and idea quality (i.e., creativity) in the innovation process.

The ideas in our model represent different solution approaches which can be ranked in terms of quality. The possibility of different approaches is also considered in Ganuza and Hauk (2006), Erat and Krishnan (2012) and Letina and Schmutzler (2019). In these papers, participants choose between a variety of approaches before exerting effort. However, as different from our model, all potential approaches are available to all solvers at the same time. In contrast, each solver has access to one and only one (different) solution approach in our model.

Another strand of literature that is related to our paper are the studies on monotone comparative statics. These studies investigate how the solutions to a maximization problem change as the parameters of the problem change. Topkis (1978) and Milgrom and Shannon (1994) consider the question in non-stochastic environments while Athey (2002) consider it in stochastic environments. Quah (2007) studies how the solution to a maximization problem changes as the constraint set changes. In this paper, we ask how the contest designer's optimal choice changes as the idea distribution changes. As the idea distribution changes, both the objective function (the seeker's expected profit), and the participants' incentive and participation constraints change.

Our stochastic order is a modified version of dominance in terms of the likelihood ratio, and is different from dominance in terms of the likelihood ratio used in Athey (2002). It is also different from the stochastic orders used in the literature to investigate how the distribution of types affect properties of equilibrium outcomes. For example, Maskin and Riley (2000) show that dominance in terms of reverse hazard rate implies more aggressive bidding in first price auctions. Pesendorfer (2000) shows similar results in procurement auctions using hazard rate dominance. Hopkins and Kornienko (2007) use dominance in terms of the likelihood ratio and a version of second order stochastic dominance to study bid aggressiveness in first-price and all-pay auctions. Hoppe et al. (2009) study the role of

[5]Some papers in the literature consider hybrid systems. For example, Fu et al. (2012) analyze how a fixed budget should be allocated between subsidies and prizes in order to motivate innovation. Galasso et al. (2018) study environments where patent rights and cash rewards are complements.

costly signaling in matching markets with privately informed agents. They use a variation of second order stochastic dominance to consider how increased heterogeneity affects the matching outcome.[6] All these papers study how equilibrium behavior changes with the distribution of private information, while we study how the mechanism designer's choice changes with the distribution of private information.

3 Model

We study a contest type which is commonly observed in innovation environments. Consider a seeker (e.g., a pharmaceutical company) who is searching for a solution (e.g., a vaccine for a new disease) and sponsors an innovation contest with a monetary prize of value $v \in [0, v_{max}]$, where v_{max} is a finite upper bound for the prize. The upper bound ensures the existence of an optimal prize. To avoid the uninteresting case in which the optimal prize is always equal to the upper bound, we also assume the bound is sufficiently large: $v_{max} > 1$.[7] The assumption that the prize is a fixed amount and does not vary with solution performance is a commonly observed feature of innovation (as well as some other type of) contests in the real world.

There is a pool of potential solvers, $\{1, 2, ..., n\}$, where $n \geq 2$. The size of this pool is fixed by the number of researchers who have the background to understand the problem posed by the contest designer (e.g., who work in the field of bioscience). Each solver i in this pool receives a private idea of quality $q_i \geq 0$, which is independently and identically distributed according to an atomless cumulative distribution function (c.d.f.) F with support $[0, w_F]$.[8] The idea quality may not be bounded, in which case $w_F = +\infty$. Different solvers receive ideas with different qualities due to differences in creativity, experience, expertise, etc. At any $q \in (0, w_F)$, F is twice continuously differentiable and its density function, $F' \equiv f$, is positive and bounded.

After receiving an idea, each solver i decides whether he would like to participate in the contest by submitting a solution. To make a submission, a solver has to develop his idea into a solution. Hence, ideas represent investment opportunities in our model.[9]

[6] The stochastic dominance they use is introduced by Barlow and Proschan (1966). For distributions with the same mean, it implies second order stochastic dominance. In contrast, the stochastic orders considered in Maskin and Riley (2000) and Pesendorfer (2000) imply first order stochastic dominance.

[7] This assumption is not important for our results. The consequence of relaxing it is discussed in Example 1.

[8] Our analysis can be generalized to $[m_F, w_F]$ with $w_F > m_F \geq 0$.

[9] Kornish and Ulrich (2014) show empirically that the quality of ideas matters in determining success. They consider "raw ideas," which are the opportunities conceived at the outset of an innovation process, and investigate the importance of having a good idea (as opposed to resources) in determining success.

While working on the problem posted by the seeker, a solver must first have an idea to invest in and then decide whether he would like to submit a solution using his idea. We assume that there is no cost for getting an idea, but a solver incurs a cost if he decides to invest in his idea.[10] Achieving a given performance level costs less if the solver has an idea of higher quality. Alternatively, with a given expenditure level, a solver with a higher quality idea produces a better solution.

Formally, if the solver's idea quality is $q_i > 0$, he can submit a solution of performance level $x_i \geq 0$ at a cost of x_i/q_i. If solver i has an idea of quality $q_i = 0$, his performance level is 0 independent of how much effort he exerts. We assume that all solvers are risk-neutral. Solver i's payoff is $v - x_i/q_i$ if he wins the prize and $-x_i/q_i$ otherwise. We use the cost function x_i/q_i for cleaner exposition, but our approach applies to a more general multiplicatively separable cost function $C(x_i, q_i) = x_i L(q_i)$ also, where L is positive, continuously differentiable, and strictly decreasing.[11]

It is reasonable to assume in innovation contests that for a solution to yield value to the seeker, its performance level must be sufficiently high. Consider, for example, the contest to develop a new vaccine. The new vaccine must have sufficiently low side effects before it can be utilized by the seeker. Hence, not all winning solutions will be of use to the seeker (see, e.g., Kremer and Glennerster, 2004). To this end, we assume that there is a publicly-known, verifiable and exogenous performance requirement or threshold $t > 0$ and solutions with performance levels below t has no benefit to the seeker. For example, the threshold may represent the minimal quality that a vaccine must satisfy for it to be approved by the regulatory authorities. We assume that $t < w_F$ to ensure that the threshold is not so high that all solvers choose zero performance.

Let $x_{(1)} = \max\{x_1, ..., x_n\}$ stand for the solution with the highest performance. If $x_{(1)} < t$, then none of the solvers wins the prize. Otherwise, the solver with the highest performance wins the prize. In case of a tie, the prize is allocated with equal probability among the tying solvers. We relax the assumption of an exogenously given threshold level in Section 6.2.

The above set-up is built on the contest model of Moldovanu and Sela (2001). The key difference is that in their model, the contest designer maximizes the total expected performance. In contrast, the seeker's profit in our set-up is a function of the maximum

[10] Hence, our modeling approach treats ideas as a primitive. This approach diverges from many R&D models in which investment opportunities are common knowledge and innovators invest to find a solution. Progress may be slow in these models if resources are scarce. We start with a model where ideas are the primitive to underline the key role they play in the innovation process. The source of the scarcity in our model is not resources, but ideas.

[11] Redefining the idea quality as $\tilde{q}_i = 1/L(q_i)$ yields the same set-up.

performance.[12] Specifically, it consists of two parts. The seeker potentially cares about obtaining a solution that is above the threshold level, and the difference between the threshold and the maximum performance:

$$\Pi(x_{(1)}, v) = \begin{cases} 1 + \lambda(x_{(1)} - t) - v & \text{if } x_{(1)} \geq t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

If the maximum performance is below the threshold, the solution generates zero profit. Any solution with a performance above the threshold t yields a fixed benefit which is normalized to 1. In addition, $\lambda \geq 0$ stands for the marginal benefit that the seeker receives from an increase in solution performance. If $\lambda = 0$, the seeker does not profit from extra performance beyond the threshold. The higher λ is, the more she benefits from an increase in performance. We study the case in which the marginal benefit is constant and relax this assumption in Section 6.1. The discontinuity in the seeker's profit function allows us to consider the case where the seeker only cares about having a solution that is above the minimum threshold. However, as we discuss in Remark 4, our analysis also applies to the case of a continuous profit function.

The seeker is assumed to be risk-neutral and chooses the prize level v to maximize her expected profit. We relax the assumption of a single prize in Section 6.3.

4 Optimal Prize

We start by deriving the Bayesian Nash equilibria in a contest with a prize of v. Since the solvers are ex ante symmetric, we focus on the symmetric equilibrium. If $v \leq t/w_F$, all solvers optimally choose zero performance. In the next lemma, we characterize the equilibrium assuming $v > t/w_F$.

Lemma 1 *In a symmetric equilibrium, a solver with idea quality q submits a solution with performance*

$$\beta(q) = \begin{cases} t + vA_t(q) & \textit{if } q \geq q_t \\ 0 & \textit{otherwise} \end{cases}$$

[12] This feature also differentiates our set-up from the all-pay auction model where the designer maximizes the expected revenue, or equivalently, the bidders' total expected payments. Our set-up is also different from other auction models. For example, in a first price auction, the seller's revenue depends on the highest bid, but only the winner pays.

where $A_t(q) = \int_{q_t}^{q} s dF^{n-1}(s)$ *and* q_t *solves*

$$q_t F^{n-1}(q_t) \;=\; t/v. \tag{2}$$

The equilibrium strategy implies that although n potential solvers receive ideas after seeing the problem posted by the contest designer, not all of them may decide to participate by submitting a solution. If a solver's idea quality is too low, he chooses zero performance. This is because the cost of submitting a solution that satisfies the threshold is higher than the expected benefit. If the idea quality is q_t, the solver is indifferent between submitting a solution of performance t and 0. If his idea quality is higher than q_t, the solver submits a solution with a performance level above t and the performance increases with his idea quality. The discontinuity of $\beta(q)$ is a result of the threshold assumption, without which the equilibrium strategy is continuous (as shown in Moldovanu and Sela, 2001).

Given the solvers' equilibrium strategy, the seeker's expected profit can be written in the following way as a function of v:

$$\Pi_F(v) = \int_{q_t}^{w_F} [1 + \lambda(\beta(q) - t) - v] dF^n(q) = L_F(v) + \lambda K_F(v) \tag{3}$$

where

$$L_F(v) \;=\; (1 - v)(1 - F^n(q_t)) \tag{4}$$

$$K_F(v) \;=\; v \int_{q_t}^{w_F} \int_{q_t}^{q} s dF^{n-1}(s) dF^n(q) \tag{5}$$

Equation (4) stands for the seeker's expected profit level when $\lambda = 0$. Since equilibrium performance level is increasing in idea quality, the probability that there is at least one solution with a performance at or above the threshold is given by $1 - F^n(q_t)$. Hence, when $\lambda = 0$, the seeker's profit is $1 - v$ with probability $1 - F^n(q_t)$ and zero otherwise. Her expected profit is $L_F(v) = (1 - v)(1 - F^n(q_t))$.

Equation (5) states the additional expected profit that the seeker makes when $\lambda > 0$. Note that the winner has the highest idea quality $q_{(1)} = \max\{q_1, ..., q_n\}$. From Lemma 1, his equilibrium performance exceeds t by $v \int_{q_t}^{q_{(1)}} s dF^{n-1}(s)$. Hence, $K_F(v)$ in equation (5) stands for the expected difference between t and the winner's performance, and $\lambda K_F(v)$ stands for the seeker's expected profit from this difference.

We consider the seeker's maximization problem. When $\lambda = 0$, the seeker maximizes $L_F(v)$. Notice that if $v \leq t/w_F \equiv \underline{v}_t$, equation (2) implies $q_t \geq w_F$. This means a solver participates only if his idea quality is w_F or higher. Since this occurs with probability zero, $L_F(v) = (1-v)(1-1) = 0$. Moreover, $L_F(v) \leq 0$ if $v \geq 1$. Thus, an optimal prize must be in $(\underline{v}_t, 1)$, which is not empty because $t < w_F$. Although the profit may not be a concave function of the prize, the following condition ensures that there is still a unique optimal prize in $(0, v_{max})$.[13]

Assumption 1 $\frac{L'_F(v)}{K'_F(v)}$ is strictly decreasing in v whenever $\frac{L'_F(v)}{K'_F(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$.[14]

Intuitively, Assumption 1 ensures that if the objective function $\Pi_F(v)$ has multiple interior local maximizers, those maximizers achieve different local maxima. If $\frac{L'_F(v)}{K'_F(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$ does not hold, then v cannot be an interior local maximizer. As we show in Appendix C, the assumption holds for many widely used parametric distribution families, such as exponential, log-normal, Pareto, power function and uniform distributions. We discuss how the results change when we relax this assumption in Section 5.3.

Proposition 1 establishes the uniqueness of the optimal prize and shows how it varies with λ for a given distribution of idea quality. If λ is sufficiently large, the optimal prize may reach the upper boundary v_{max}. Let $\bar{\lambda}_F$ stand for the smallest value of λ such that v_{max} is an optimal prize. Recall that the optimal prize is below 1 if $\lambda = 0$, so $\bar{\lambda}_F > 0$.

Proposition 1 *Under Assumption 1, there is a unique optimal prize $V_F(\lambda)$ for all marginal benefits $\lambda \neq \bar{\lambda}_F$. Moreover, $V_F(\lambda)$ is weakly increasing in λ.*

Proposition 1 states that as the marginal benefit of solution performance to the seeker increases, the seeker finds it optimal to offer a higher prize. Offering a higher prize encourages the solvers to submit solutions with higher performance levels.

When $\lambda = \bar{\lambda}_F$, there may be two optimal prizes: an interior prize in $(0, v_{max})$ and v_{max}. Then, let $V_F(\bar{\lambda}_F)$ be either one of the optimal prizes. Our results do not depend on the choice for $V_F(\bar{\lambda}_F)$. When there are two optimal prizes, $V_F(\lambda)$ is not continuous in λ.

5 Scarcity of Ideas

In this section, we study comparative statics of the optimal prize with respect to the scarcity of high-quality ideas for a given value of λ.

[13]For example, if $F(q) = q$ and $t = 1/2$, $K_F(v)$ is convex. Thus, for sufficiently large λ, the profit $L_F(v) + \lambda K_F(v)$ is not concave.

[14]See (A.58) for the expression of $\lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$.

5.1 Stochastic Dominance Notion of Idea Scarcity

Different challenges in our model are represented by different idea distributions. In easy challenges, many innovators are likely to have high-quality ideas to start with. As challenges get more difficult, the probability of drawing a high-quality idea decreases.

Intuitively, the concept of idea scarcity depends on both the difficulty of the challenge and the number of solvers. In particular, for a given number of solvers, as the difficulty level of a challenge increases, high-quality ideas would become scarcer. Similarly, in a challenge with a given level of difficulty, as the number of potential solvers decreases, high-quality ideas would become scarcer.

To capture both of these contributing factors in our definition of idea scarcity, we first define "effective quality" of a solver's idea. We then use it to introduce a stochastic dominance notion of idea scarcity.

Definition 1 If a solver has an idea of quality q drawn from distribution F, we define *effective quality* of his idea as $\phi_F(q) \equiv qF^{n-1}(q)$. Then, the c.d.f. of the effective quality, $\hat{F} : [0, w_F] \to [0, 1]$, takes the form $\hat{F}(x) \equiv F(\phi_F^{-1}(x))$. We refer to $\hat{F}$ as the effective distribution of F.

For a solver with idea q_i, his effective quality is q_i discounted by $F^{n-1}(q_i)$, which is the probability that his idea quality is higher than that of all other solvers. Hence, the concept of effective quality, which is crucial for our analysis below, contains information on both a solver's marginal cost, through q_i, and his probability of winning, through $F^{n-1}(q_i)$. Both pieces of information are important for a solver's decision. In comparison, q_i and its distribution F, or $q_{(1)}$ and its distribution F^n contain only one piece of information.

Remark 1 In auction theory, the expected surplus from trading with a bidder is defined in a similar way to the effective quality defined in Definition 1 (e.g., Bulow and Roberts, 1989). To see this, consider a first-price or second-price auction with n symmetric bidders whose values are i.i.d. according to a c.d.f. F. If the seller trades with a bidder with value v, the total surplus is v. In equilibrium, trade happens with probability $F^{n-1}(v)$, which represents the probability that the bidder's value is higher than the values of the other bidders. Therefore, the expected surplus from trading with a bidder is $x = vF^{n-1}(v)$ and its c.d.f. is $\hat{F}$.

The effective distribution preserves important properties of the original distribution, such as first order stochastic dominance (FOSD) and log-concavity:

Lemma 2 *Distribution G first order stochastically dominates F, written as $F \prec_{FOSD} G$, if and only if $\hat{F} \prec_{FOSD} \hat{G}$.*

Lemma 3 *Suppose $q + F(q)/f(q)$ is non-decreasing. Then, if F is log-concave, $\hat{F}$ is also log-concave.*

The assumption of non-decreasing $q + F(q)/f(q)$ is often used in mechanism design, and it ensures the virtual cost $q + F(q)/f(q)$ of a seller is non-decreasing in his cost q (see, e.g., Myerson and Satterthwaite, 1983). Notice that $q + F(q)/f(q)$ is non-decreasing if F is log-concave, which implies that this assumption is less restrictive than log-concavity.[15]

We now introduce a stochastic dominance order in order to rank idea scarcity.

Definition 2 A distribution $\hat{G}$ dominates $\hat{F}$ in terms of the likelihood ratio, written as $\hat{F} \prec_{LR} \hat{G}$, if

$$\frac{\hat{g}(x)}{\hat{f}(x)} \leq \frac{\hat{g}(x')}{\hat{f}(x')}$$

for any x, x' in the union of $\hat{G}$ and $\hat{F}$'s supports such that $x < x'$. We say F represents scarcer ideas than G, written as $F \prec G$, if $\hat{F} \prec_{LR} \hat{G}$.

It is important to emphasize that the difficulty of a challenge is a different but related concept to the scarcity of ideas. The difficulty of a challenge is independent of the size of the solver base (n), and is captured by the idea distribution F. If the challenge is a difficult one, then the probability of having a high-quality idea decreases. In contrast, a solver has a scarcer idea if the probability that someone else has a better idea than him is smaller. Since this probability depends on the size of the solver base (n), the scarcity order depends on n as well.

Note that the above scarcity order is different from dominance in terms of the likelihood ratio (see, e.g., Milgrom, 1981), which requires $\frac{g(q)}{f(q)}$ to be increasing in q over the union of F and G's supports. The above stochastic dominance order is defined indirectly using effective distributions. This approach is similar to the definition of (reverse) hazard rate dominance, which is specified using (reverse) hazard rates. Reverse hazard rate dominance and hazard rate dominance are widely used in the literature on auctions. We need a different stochastic order concept because of the all-pay feature of our design and the fact that the seeker's profit depends on maximum performance instead of total performance.

[15] See Bagnoli and Bergstrom (2005) for a survey of applications of log-concave distributions. They also provide a comprehensive list of parametric distributions that are log-concave.

The following result discusses how the stochastic order introduced in Definition 2 relates to FOSD. As in the case of hazard rate or reverse hazard rate dominance, the stochastic order stated in Definition 2 is stronger than FOSD. However, as we show in Appendix D, it is equivalent to FOSD for many widely used parametric distribution families, such as exponential, log-normal, Pareto, power function and uniform distributions.

Lemma 4 *$F \prec G$ implies $F \prec_{FOSD} G$.*

It is worth noting that the comparison of two distributions in terms of scarcity of ideas depends on n, the (exogenously given) number of potential solvers. The following result shows that the scarcity order of two distributions is never reversed as n varies.

Lemma 5 *If $F \prec G$ for some $n \in \mathbb{N}$, there is no $n' \in \mathbb{N}$ such that $G \prec F$.*

To see why, suppose $F \prec G$ for n and $G \prec F$ for n'. Then, Lemma 4 implies $F \prec_{FOSD} G$ and $G \prec_{FOSD} F$ which cannot happen.

We end this section by showing the relationship between our scarcity order and two observable characteristics of innovation contests. Recall that (2) implies that $\phi_F^{-1}(t/v) = q_t$, so $1-\hat{F}(t/v) = 1-F(q_t)$ is the ex ante probability for a solver to actively participate (i.e., submit a solution with positive performance) in a contest. We refer to $P_F(v) \equiv 1-\hat{F}(t/v)$ as the *participation rate* and $1 - P_F(v)$ as the *non-participation rate*. Then, the expected number of actively participating players is $nP_F(v)$. The elasticity of the participation rate w.r.t. the prize is $vP'_F(v)/P_F(v)$, and the elasticity of the non-participation rate w.r.t. the prize is $-vP'_F(v)/(1 - P_F(v))$.

The following lemma shows that scarcer ideas lead to more elastic participation and non-participation rates. Intuitively, when high-quality ideas are scarcer, there is higher population density at lower idea qualities, especially around the critical quality between participation and non-participation. As a result, a marginal change in prize has a larger effect on participation.

Lemma 6 *$F \prec G$ implies that for any $v > 0$,*

$$P'_G(v)/P_G(v) \quad < \quad P'_F(v)/P_F(v) \tag{6}$$

$$-P'_G(v)/(1 - P_G(v)) \quad < \quad -P'_F(v)/(1 - P_F(v)) \tag{7}$$

The proof is straightforward from the definitions and is therefore omitted. Note that all our results remain the same if we replace $F \prec G$ with (6) and (7). This is discussed further in Remark 3 below. Lemma 6 implies that if a seeker could obtain information, for instance through a survey, on the participation rates of two challenges at different prize levels, she could compare the scarcity of ideas between the two challenges. In the challenge with scarcer ideas, the participation rate would be lower at all prize levels.

5.2 Comparative Static Analysis

Consider two continuously differentiable distributions F and G, whose supports $[0, w_F]$ and $[0, w_G]$ may be different. Assume that the density functions $F' \equiv f$ and $G' \equiv g$ are positive. Recall that $L_F(v)$ and $K_F(v)$ described in (4) and (5) only depend on F. The counterparts for distribution G, $L_G(v)$ and $K_G(v)$, similarly only depend on G. Moreover, similar to (2), the marginal solver under distribution G, who is indifferent between participating and not participating, has idea quality q_t' defined by

$$q_t' G^{n-1}(q_t') = t/v \tag{8}$$

The assumption below ensures that $V_F(\lambda)$ and $V_G(\lambda)$ cross exactly once.[16]

Assumption 2 $\frac{L_F'(v)}{K_F'(v)}$ and $\frac{L_G'(v)}{K_G'(v)}$ cross at most once over the range of v such that $\frac{L_F'(v)}{K_F'(v)} > \lim_{v\to\infty} \frac{L_F'(v)}{K_F'(v)}$ and $\frac{L_G'(v)}{K_G'(v)} > \lim_{v\to\infty} \frac{L_G'(v)}{K_G'(v)}$.

As we show in Appendix C, the assumption holds for many widely used parametric distribution families, such as exponential, log-normal, Pareto, power function and uniform distributions.

The following proposition is the main result of the paper. We present its proof in Appendix B.

Proposition 2 *Under Assumptions 1 and 2, if $F \prec G$, there exists a unique $\hat{\lambda} > 0$ such that*

(i) $V_G(\lambda) < V_F(\lambda)$ if $\lambda < \hat{\lambda}$;

(ii) $V_G(\lambda) \geq V_F(\lambda)$ if $\lambda > \hat{\lambda}$.

That is, scarcer ideas lead to a lower optimal prize if and only if the marginal benefit of solution performance is sufficiently high.

It is worth mentioning that when $\lambda = \hat{\lambda}$, it is possible that $V_G(\hat{\lambda}) = V_F(\hat{\lambda})$, $V_G(\hat{\lambda}) < V_F(\hat{\lambda})$ or $V_G(\hat{\lambda}) > V_F(\hat{\lambda})$. In the last two cases, $V_G(\lambda)$ may not change continuously at $\hat{\lambda}$, and it may jump to v_{max} if λ is slightly above $\hat{\lambda}$.

[16] In general, $V_F(\lambda) - V_G(\lambda)$ may not be monotone in λ. For example, Figure 1 shows that $V_F(\lambda) - V_G(\lambda)$ is decreasing for small values of λ and increasing for large values of λ.

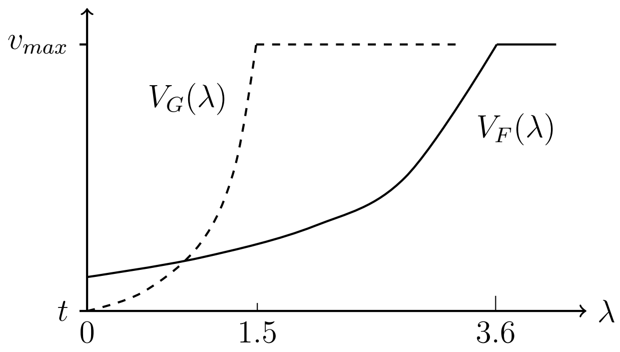

Figure 1: Optimal Prizes

Proposition 2 implies that the value of λ plays a critical role in determining how the optimal prize changes with the distribution of ideas. We first illustrate Proposition 2 with the following example and then explain the intuition behind it.

Example 1 *Suppose* $t = 0.4$*,* $n = 4$*,* $v_{max} = 1.5$ *and consider two Pareto distributions* $F(q) = 1 - (1+q)^{-4}$ *and* $G(q) = 1 - (1+q)^{-2}$*. The distributions satisfy Assumptions 1 and 2 and* $F \prec G$*. As in Figure 1, there exists* $\hat{\lambda} \approx 0.9$ *such that* $V_F(\lambda) > V_G(\lambda)$ *if* $\lambda \in [0, \hat{\lambda})$*,* $V_F(\lambda) = V_G(\lambda)$ *if* $\lambda = \hat{\lambda}$*, and* $V_F(\lambda) < V_G(\lambda)$ *if* $\lambda \in (\hat{\lambda}, 3.6)$*. For* $\lambda \geq 3.6$*,* $V_F(\lambda) = V_G(\lambda) = v_{max}$*.*[17]

For the intuition behind the result, consider the seeker's expected payoff function given in (3). When $\lambda = 0$, the seeker maximizes (4). In this case, what is important is to have a solution with performance above the minimum threshold. As the distribution of ideas shifts such that high-quality ideas become scarcer, it puts downward pressure on the probability of success (i.e., the probability that at least one solution meets the threshold). In addition, scarcer ideas reduce competition among the solvers and therefore increase participation, which puts upward pressure on the probability of success. As it turns out, the first effect always dominates the second one; that is, scarcer ideas lead to a lower probability of success. Hence, the seeker compensates by increasing the prize level. On the other hand, when high-quality ideas are abundant, the probability of success is high to start with, so the marginal return of the prize is lower and the seeker does not need to have as high a prize.

For a graphical illustration, when $\lambda = 0$, consider the marginal profit functions $L'_F(v)$ and $L'_G(v)$ in Figure 2. As shown in Lemma A.1 in Appendix B, the marginal profit function under more abundant ideas, $L'_G(v)$, reaches the horizontal axis before the marginal

[17]Recall that we assume $v_{max} > 1$. If v_{max} is too small, for instance $v_{max} < 0.4$, the optimal prize may be forced to be the upper bound: $V_G(\lambda) = V_F(\lambda) = v_{max}$ for all λ. Moreover, if $v_{max} = V_G(\hat{\lambda}) = V_G(\hat{\lambda})$, then $V_F(\lambda)$ and $V_G(\lambda)$ are forced to be v_{max} for $\lambda > \hat{\lambda}$. In this case, Proposition 2 still holds, but the strict inequality $V_G(\lambda) > V_F(\lambda)$ illustrated in Example 1 may not arise.

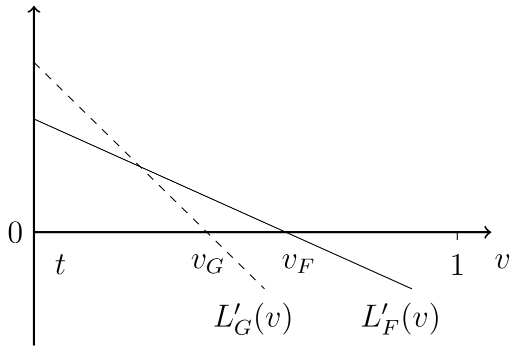

Figure 2: Optimal Prizes if $\lambda = 0$

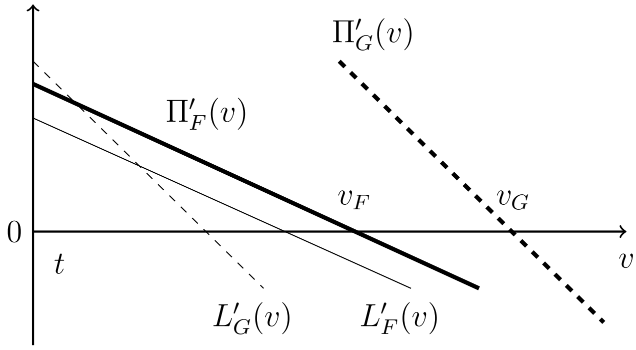

Figure 3: Optimal Prizes if λ is Large

profit function under scarcer ideas, $L'_F(v)$, does. Since the unique optimal prize satisfies the first order conditions $L'_F(v) = 0$ and $L'_G(v) = 0$, this implies that $V_F(\lambda) > V_G(\lambda)$. That is, as high-quality ideas become scarcer, the seeker optimally increases the prize.

Consider next $\lambda > 0$. The seeker maximizes the sum of (4) and (5). That is, the seeker cares both about meeting the threshold and receiving as good a solution as possible. The magnitude of λ determines how much weight the seeker will put on (5). If the seeker increases the prize, more solvers participate and each participating solver increases his performance. Thus, an increase in the prize pushes the expected performance further beyond the threshold. When ideas are abundant, the average idea quality (solvers' productivity) is higher, so the seeker's marginal return of the prize is higher. Hence, if λ is sufficiently high, then this effect dominates and the seeker finds it optimal to set a higher (lower) prize in challenges where ideas are more abundant (scarce).

This is demonstrated in Figure 3, which illustrates two properties of K'_F and K'_G established in Lemma A.2 in Appendix B. In the figure, $\lambda K'_F(v)$ corresponds to the vertical distance between the curves $\Pi'_F(v)$ and $L'_F(v)$. The first property stated in Lemma A.2 is that the expected additional performance beyond the threshold is increasing in the prize; that is, $K'_F(v) > 0$. Then, a positive λ shifts $L'_F(v)$ upwards to be $\Pi'_F(v) = L'_F(v) + \lambda K'_F(v)$. Moreover, because $\lambda K'_F(v)$ is larger with a higher λ, $V_F(\lambda') \geq V_F(\lambda)$ if $\lambda' \geq \lambda$. In other words, because a larger λ leads to a higher marginal profit of prize, the optimal prize should not be lower.

The second property stated in Lemma A.2 implies that $K_F(v)$ is supermodular in $(v; F)$, which means the marginal impact of the prize on $K_F(v)$ is smaller when ideas are scarcer. In Figure 3, the distance between $L'_G(v)$ and $\Pi'_G(v)$ is larger than the distance between $L'_F(v)$ and $\Pi'_F(v)$. As a result, if λ is sufficiently large, $\Pi'_G(v)$ is larger than $\Pi'_F(v)$.

In summary, when $\lambda > 0$, a marginal increase in the prize amount pushes the expected performance further beyond the minimal threshold. Moreover, the marginal increase in the prize amount has a higher return if the solvers have more abundant ideas. As a result, as high-quality ideas become more abundant, the seeker optimally increases the prize.

The above discussion interestingly uncovers that while maximizing the expected payoff function $\Pi_F(v) = L_F(v) + \lambda K_F(v)$, idea abundance and prizes play substitute roles in the first term and complementary roles in the second term. This represents the main trade-off faced by a contest designer and λ determines the relative weight of the two elements.

We end this section with three remarks related to Proposition 2. Remark 2 explains how our comparative static result differs from the literature.

Remark 2 First, many comparative static results in the literature rely on submodularity or supermodularity (e.g. Topkis, 1978). In this paper, the objective function $L_F(v)$ is neither submodular nor supermodular in $(v; F)$. Specifically, notice that the submodularity of $L_F(v)$ in $(v; F)$ requires $L'_G(v) < L'_F(v)$ for all v, but Lemmas A.7 and A.9 in Appendix E show otherwise (as illustrated in Figure 2).[18] Second, other comparative static results in the literature rely on single-crossing conditions (e.g., Milgrom and Shannon, 1994).[19] The distinguishing feature of our comparative static analysis is that it is with respect to distributions, and Lemma A.1 in Appendix B states a functional form of the single-crossing condition for $L_F(v)$ in $(v; F)$.

For cleaner exposition, we use the stochastic order defined in Definition 2. However, the next remark explains how our results apply to a more general stochastic order.

Remark 3 It is well-known that dominance in terms of both the hazard rate and the reverse hazard rate is less restrictive than dominance in terms of the likelihood ratio.[20] All of our results and their proofs remain unchanged if in Definition 2, instead of saying $\hat{F} \prec_{LR} \hat{G}$, we say $\hat{G}$ dominates $\hat{F}$ in terms of the hazard rate and in terms of the reverse hazard rate. We can also relax $F \prec G$ to (6) and (7). This is because the two inequalities imply that $\hat{G}$ stochastically dominates $\hat{F}$ in terms of reverse hazard rate and hazard rate, which is sufficient for all our results.

The seeker's profit in (1) has a discontinuous jump at the threshold, which arises if meeting the threshold results in a non-trivial profit. The remark below shows that our analysis applies even if the seeker's profit is continuous.

[18] Lemma A.7 implies $L'_F(1) - L'_G(1) > 0$ and Lemma A.9 implies $L'_F(t) - L'_G(t) < 0$ if F and G have a common support $[0, 1]$. Therefore, $L'_F(v)$ and $L'_G(v)$ must intersect as in Figure 2.

[19] They also require quasi-submodularity of the objective function.

[20] See, for instance, Shaked and Shanthikumar (2007), p. 55.

Remark 4 Suppose that if the maximum performance is above the minimum performance threshold, the seeker's profit is given by

$$\Pi(x_{(1)}, v) = \begin{cases} \lambda(x_{(1)} - t) - v & \text{if } x_{(1)} \geq t \\ 0 & \text{otherwise} \end{cases}$$

instead of (1). That is, the gross profit, $\max(\lambda(x_{(1)} - t), 0)$, is continuous in the maximum performance. Then, $L_F(v) = -v(1 - F^n(q_t))$ and $K_F(v)$ remain the same. Since there is no profit obtained from reaching the minimum performance threshold, if the marginal benefit is $\lambda = 0$, then $V_F(\lambda) = V_G(\lambda) = 0$. By virtually the same proof as Proposition 2, we obtain the critical value $\hat{\lambda} = 0$ and case (i) in Proposition 2 never arises. This means that in this case, scarcer ideas always lead to lower prizes.

5.3 A Generalization of Proposition 2

We next consider a generalization of the result stated in Proposition 2 by relaxing Assumptions 1 and 2. Without Assumption 1, there may be multiple optimal prizes. Then, let $V_F(\lambda)$ and $V_G(\lambda)$ represent the set of optimal prizes. To compare sets of possibly multiple optimal prizes, we use the strong set order (see, e.g., Topkis, 1978).

Definition 3 For two sets $A, B \subset \mathbb{R}^m$, A is greater than B in the strong set order, written as $A \geq B$, if, for any $a \in A$ and any $b \in B$, the pointwise maximum $a \vee b \in A$ and the pointwise minimum $a \wedge b \in B$.

If $m = 1$, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For example, $\{2, 3, 4\} \geq \{1, 2, 3\}$ and $[2, 4] \geq [1, 3]$. If both A and B are singletons, the strong set order reduces to the order over real numbers. The following proposition shows that a result similar to Proposition 2 can be obtained if Assumptions 1 and 2 are relaxed.

Proposition 3 *If $F \prec G$, there exist $\hat{\lambda}' \geq \hat{\lambda} > 0$ such that*

i) $V_F(\lambda) \geq V_G(\lambda)$ if $\lambda < \hat{\lambda}$.

ii) $V_F(\lambda) \leq V_G(\lambda)$ if $\lambda > \hat{\lambda}'$.

Notice that $\hat{\lambda}'$ and $\hat{\lambda}$ may be different, which is the key difference between Propositions 2 and 3. This is partly because the strong set order is only a partial order if $V_F(\lambda)$ or $V_G(\lambda)$ contains multiple prizes. In contrast, when the optimal prize is unique as in Proposition 2, the strong set order becomes the complete order over real numbers.

6 Extensions

6.1 Nonlinear Benefits

So far we have assumed that the marginal benefit of additional performance above the threshold is constant. In this section, we show that our results continue to hold when we relax this assumption by generalizing the seeker's benefit to

$$\Pi(x_{(1)}) = \begin{cases} 1 + \lambda B(x_{(1)} - t) - v & \text{if } x_{(1)} \geq t \\ 0 & \text{otherwise} \end{cases}$$

where B is differentiable and satisfies $B(0) = 0$, $B'(x) > 0$ for all $x \geq 0$. It is not necessarily concave or convex. In previous sections, we considered the special case where $B(x) = x$.

Since the change in the seeker's benefit function does not have an impact on the solvers' behavior, the solvers' equilibrium strategies remain the same as in Lemma 1. As in Section 3, we can also write the expected profit as $\Pi_{F,B}(v) = L_F(v) + \lambda K_{F,B}(v)$, where $L_F(v)$ is the same as in (4) and

$$K_{F,B}(v) = \int_{q_t}^{w_F} B\left(v \int_{q_t}^{q} s dF^{n-1}(s)\right) dF^n(q)$$

is the expected benefit from additional performance above the threshold t.

We generalize Assumptions 1 and 2 to nonlinear benefit functions as follows:

Assumption 1′ $\frac{L'_F(v)}{K'_{F,B}(v)}$ is strictly decreasing in v whenever $\frac{L'_F(v)}{K'_{F,B}(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_{F,B}(v)}$.

Assumption 2′ $\frac{L'_F(v)}{K'_{F,B}(v)}$ and $\frac{L'_G(v)}{K'_{G,B}(v)}$ cross at most once over the range of v such that $\frac{L'_F(v)}{K'_{B,F}(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_{B,F}(v)}$ and $\frac{L'_G(v)}{K'_{B,G}(v)} > \lim_{v\to\infty} \frac{L'_G(v)}{K'_{B,G}(v)}$, and $\lim_{x\to\infty} B'(x) = \underline{b} > 0$.

The first part of Assumption 2′ is similar to Assumption 2, but the second part is new. This part ensures the marginal value $B'(x)$ converges to a positive value as x goes to infinity. If this is violated and the marginal value converges to zero, for sufficiently large λ, the comparative statics are similar to $\lambda = 0$. This is because the benefit function for sufficiently large λ has a similar shape to that with $\lambda = 0$.

Replacing Assumptions 1 and 2 with Assumptions 1′ and 2′, we can generalize Propositions 1-3 to accommodate nonlinear benefits. This is done in Propositions A.1-A.3. Because these results and their proofs are similar to those of Propositions 1-3, we relegate them to the appendix.

6.2 Endogenous Reservation Performance

In this section, we assume that the seeker can choose both a prize value v and a performance level r such that the prize is awarded only if the highest performance is at least r. Since r plays a similar role to a reservation price in the auction literature, we call it the reservation performance.

Note that given the seeker's choice (v, r), the solvers' equilibrium strategy is the same as in Lemma 1 except that t is replaced by r. An increase in r causes an upward shift in $\beta(q_i)$. Hence, the seeker may want to set a higher r in order to elicit solutions with higher performance levels, which becomes more important as λ increases.

As we show in Lemma A.11 in Appendix G, although the seeker may choose any $r \geq 0$, choosing a reservation performance level $r < t$ is never optimal. This is because performance levels below t are worthless to the seeker. Thus, we only need to consider $r \geq t$. Given such a $r \geq t$, solvers with $q_i < q_r$ do not participate and those with $q_i > q_r$ choose a performance level above $r \geq t$, where q_r is the unique solution of

$$q_r F^{n-1}(q_r) = r/v. \tag{9}$$

As a result, the expected profit of the seeker is given by

$$\int_{q_r}^{w_F} [1 + \lambda(\beta(q) - t)] dF^n(q) - v(1 - F^n(q_r))$$

Substituting $\beta(q)$ into the profit, we can rewrite it as a function of v and q_r:[21]

$$\Pi_F(v, q_r) = L_F(v, q_r) + \lambda K_F(v, q_r) \tag{10}$$

where

$$\begin{aligned} L_F(v, q_r) &= (1 - v)(1 - F^n(q_r)) \\ K_F(v, q_r) &= \int_{q_r}^{w_F} \left[vF^{n-1}(q) \left(q - \frac{1 - F^n(q)}{(F^n)'(q)} \right) - t \right] dF^n(q) \end{aligned}$$

Hence, the seeker chooses (v, q_r) instead of (v, r). Since $r \geq t$ is equivalent to $q_r \geq q_t$, the seeker chooses $v \geq 0$ and $q_r \geq q_t$ to maximize $\Pi_F(v, q_r)$. Given (v, q_r), r can be recovered from (9). Let $VR_F(\lambda)$ denote the set of optimal pairs of $(v, r) \in [0, v_{max}] \times \mathbb{R}_+$ when the idea quality distribution is F.

[21] For more details, see Appendix G.

We show in Lemma A.12 that when $\lambda = 0$, even though the seeker has the option to choose a r value that is different from t, it is optimal to choose $r = t$. We already know that $r < t$ is not optimal. It is not optimal to announce $r > t$ either since it causes the seeker to miss on profitable opportunities.

However, for $\lambda > 0$, there may be a reason to set $r > t$. This is because setting $r > t$ may increase performance further beyond the threshold t, which becomes important for sufficiently high values of λ. Lemma A.13 formalizes this intuition.

The next assumption ensures that for all λ, each solver's probability of participation is bounded away from zero under the optimal reservation performance and prize.[22]

Assumption 3 $q - \dfrac{1 - F^n(q)}{(F^n)'(q)}$ increases in q, and $\lim_{q \to w_F} \left(q - \dfrac{1 - F^n(q)}{(F^n)'(q)} \right) > 0$.[23]

Assumption 3 states that the virtual valuation of $q_{(1)}$, the best idea quality, is increasing, and it is positive at the highest value in the support.

In order to compare $VR_F(\lambda)$ for different λ, we need to use the strong set order given in Definition 3 for higher dimensions with $m = 2$. We can also use the order to define monotonicity of $VR_F(\lambda)$ and $VR_G(\lambda)$. For example, we say $VR_F(\lambda)$ is monotone non-decreasing/non-increasing in λ if for any $\lambda_1 > \lambda_2$, $VR_F(\lambda_1)$ is no lower/higher than $VR_F(\lambda_2)$. The following result is analogous to Proposition 1.

Proposition 4 *Under Assumption 3, there exist $\lambda'' \geq \lambda' > 0$ such that*

i) $VR_F(\lambda)$ is monotone non-decreasing for $\lambda < \lambda'$.

ii) $VR_F(\lambda)$ is monotone non-increasing for $\lambda > \lambda''$.

As different from Proposition 1, Proposition 4 considers both the optimal prize and the optimal reservation performance, but it focuses on sufficiently small or large λ values only. In the proof of Proposition 4, we show that for λ sufficiently small, the optimal reservation performance is equal to the threshold, and the optimal prize is increasing in λ. In contrast, for sufficiently large λ, the optimal prize is equal to the upper boundary v_{max}, and the optimal reservation performance is strictly decreasing in λ. Hence, the optimal prize is weakly increasing in λ for $\lambda < \lambda'$ and $\lambda > \lambda''$, which is in line with Proposition 1.

The intuition for the above result is as follows. Notice that $\frac{\partial^2 L_F(v, q_r)}{\partial v \partial q_r} > 0$, and q_r is strictly increasing in r due to (9), so v and r play complementary roles in $L_F(v, q_r)$. In

[22] See the proof of Proposition 4 in Appendix G.

[23] This assumption allows for the case where $\lim_{q \to w_F} \left(q - \dfrac{1 - F^n(q)}{(F^n)'(q)} \right) = +\infty$.

contrast, $\frac{\partial^2 K_F(v,q_r)}{\partial v \partial q_r} < 0$, which means v and q_r play substitute roles in $K_F(v, q_r)$.[24] As a result, as λ increases from 0, the optimal v and q_r first change in the same direction due to their impact on $L_F(v, q_r)$, but eventually they may move in different directions due to their impact on $K_F(v, q_r)$.

For a given λ, let $VR_G(\lambda)$ be the set of optimal (v, r) associated with distribution G. Recall that $r = 0$ if $\lambda = 0$. Therefore, Proposition 2 implies that $VR_F(\lambda) \geq VR_G(\lambda)$ if $\lambda = 0$. The comparison of $VR_F(\lambda)$ and $VR_G(\lambda)$ for sufficiently large λ is closely related to the two equations below:

$$F^{n-1}(q)\left(q - \frac{1 - F^n(q)}{(F^n)'(q)}\right) = \frac{t}{v_{max}} \tag{11}$$

$$G^{n-1}(q)\left(q - \frac{1 - G^n(q)}{(G^n)'(q)}\right) = \frac{t}{v_{max}} \tag{12}$$

Equation (11) is the first order condition $\frac{\partial \Pi_F(v,q_r)}{\partial q_r} = 0$ evaluated at $\lambda \to \infty$, and (12) is the counterpart for G.[25] These equations have a unique solution under Assumption 3. Let q_F be the solution to (11) and q_G be the solution to (12). The following result compares $VR_F(\lambda)$ and $VR_G(\lambda)$:

Proposition 5 *Under Assumption* 3*, if* $F \prec G$*, there exist* $\hat{\lambda}' \geq \hat{\lambda} > 0$ *such that*

i) for $\lambda < \hat{\lambda}$, $\quad VR_F(\lambda) \geq VR_G(\lambda)$.

ii) for $\lambda > \hat{\lambda}'$, $\quad VR_F(\lambda) \leq VR_G(\lambda)$ *if* $q_G G^{n-1}(q_G) \geq q_F F^{n-1}(q_F)$;

$\quad VR_F(\lambda) \geq VR_G(\lambda)$ *if* $q_G G^{n-1}(q_G) < q_F F^{n-1}(q_F)$.

Proposition 5 is analogous to Proposition 3. We show in the proof of Proposition 5 that the optimal reservation performance is equal to the threshold for $\lambda < \hat{\lambda}$. Hence, part i) of Proposition 5 generalizes part i) of Proposition 3 by considering both the optimal prize and reservation performance. The difference in part ii) between Propositions 3 and 5 comes from the optimal reservation performance. As we show in the proof, the optimal reservation performance with G, representing the case of more abundant ideas, will be lower if the corresponding effective quality of q_G is lower than that of q_F.

The results in Proposition 5 follow from the following observations made in the proof. When λ is sufficiently small, the optimal prize associated with F is higher than that associated with G. The optimal reservation performance under both F and G is the same and equal to r. When λ is sufficiently large, the optimal prize under both F and G

[24] Notice that $K_F(v, q_r) = \int_{q_r}^{w_F} [\beta(q) - t]\, dF^n(q) = v \int_{q_r}^{w_F} \int_{q_r}^{q} s dF^{n-1}(s) dF^n(q)$, so $\frac{\partial^2 K_F(v,q_r)}{\partial v \partial q_r} = -q_r (F^{n-1})'(q_r)(1 - F^n(q_r)) < 0$.

[25] See the proof of Proposition 5 in Appendix G.

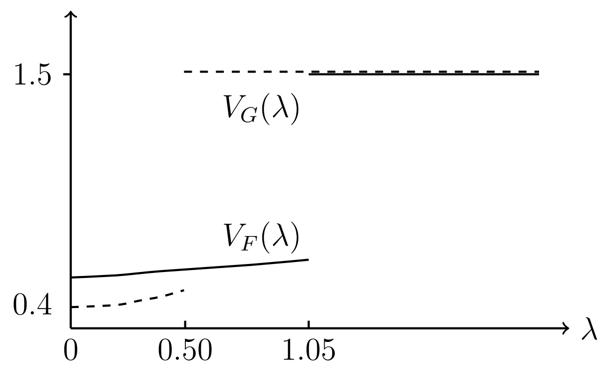

Figure 4: Optimal Prize

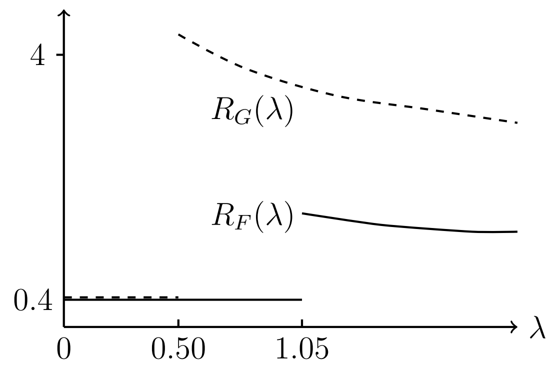

Figure 5: Optimal Reservation Performance

is equal to the upper boundary v_{max}. If $\lambda \to +\infty$, the optimal reservation performance associated with F is above r and converges to $v_{max} q_F F^{n-1}(q_F)$ and that associated with G is above r and converges to $v_{max} q_G G^{n-1}(q_G)$.

The following example illustrates Proposition 5.

Example 2 *Consider Example 1, but now assume that the seeker chooses (v, r). Recall that $t = 0.4$, $n = 4$, $v_{max} = 1.5$. For each λ, there is a unique optimal prize, denoted as $V_F(\lambda)$, and a unique optimal reservation performance, denoted as $R_F(\lambda)$. There exists $\hat{\lambda} = 0.50$ such that $V_F(\lambda) > V_G(\lambda)$ and $R_F(\lambda) = R_G(\lambda)$ for $\lambda < \hat{\lambda}$, and $V_G(\lambda) \geq V_F(\lambda)$ and $R_G(\lambda) > R_F(\lambda)$ for $\lambda > \hat{\lambda}$. Figures 4 illustrates the optimal prizes and Figure 5 illustrates the optimal reservation performance.*

6.3 Winner-Take-All vs. Multiple Prizes

So far we have been focusing on a winner-take-all prize structure. We next generalize our results to a setup without the restriction of a single prize. Specifically, suppose that $n \geq 3$ and the seeker chooses at most two prizes $v^1 \geq v^2 \geq 0$. If the two highest performing solutions, $x_{(1)}$ and $x_{(2)}$, reach the threshold t, the solution with the highest performance receives the highest prize v^1 and the solution with the second highest performance receives v^2. If only $x_{(1)}$ reaches the threshold, then v^1 is awarded to the winner and v^2 is not awarded. If neither $x_{(1)}$ nor $x_{(2)}$ reaches the threshold, no prizes are awarded.

Multiple prizes bring several complications to the seeker's choices: First, the seeker needs to choose both the size of the purse (total budget) and the allocation of the purse between the prizes. This differs from the large and growing literature on how to optimally allocate a given budget between prizes (e.g., Krishna and Morgan, 1998 and Moldovanu and Sela, 2001). The second complication is that multiple prizes may help the seeker reduce expenditure in prizes. To see this, notice that if only $x_{(1)}$ reaches the threshold, the seeker only awards the first prize and saves the second. When the contest designer must spend a given budget, this effect does not arise.

Let $\bar{v}$ stand for the budget for the prizes. She allocates the purse into v^1 and v^2 with $v^1 \geq v^2 \geq 0$. Let $\bar{V}_F(\lambda)$ be the set of optimal budgets for the distribution F and marginal benefit λ. The result below generalizes Proposition 3 to multiple prizes.

Proposition 6 *Suppose 1) $\hat{G}$ or $\hat{F}$ has increasing hazard rate, 2) $\hat{G}$ or $\hat{F}$ is log-concave.*[26] *Then, for $F \prec G$, there exist $\lambda'' \geq \lambda' > 0$ such that*

i) for $\lambda < \lambda'$, two equal prizes are optimal
for $\lambda > \lambda''$, winner-take-all is optimal;

ii) for $\lambda < \lambda'$, scarcer ideas lead to larger optimal purses: $\bar{V}_G(\lambda) \leq \bar{V}_F(\lambda)$,
for $\lambda > \lambda''$, scarcer ideas lead to smaller optimal purses: $\bar{V}_F(\lambda) \leq \bar{V}_G(\lambda)$.

The above result demonstrates a novel characteristic of contests that affects the optimal allocation of prizes: the marginal benefit λ.[27] This is because multiple prizes encourage solvers with lower quality ideas to participate. If λ is small, multiple prizes are optimal because more participations lead to higher probability of success. As λ becomes larger, by how much the highest performance exceeds the threshold is more important for the seeker's profit than the probability of success. Since the highest performance is unlikely to be achieved by those with lower idea qualities, the seeker prefers to merge the multiple prizes into a single one to provide stronger incentive to the solvers with higher idea qualities.

7 Conclusion

Although innovation contests have a long history, there has been an increase in their use in recent years. One of the reasons for this growth in popularity of innovation contests is that progress in information technology has made it easier to run innovation contests using the Internet. As a result, several innovation platforms have emerged on the Internet as the meeting place of seekers of innovative solutions and solvers of innovation problems.[28] Especially when it is not possible to identify ex ante who has the expertise to solve a specific challenge, it is useful to make the challenge public to many potential solvers.[29]

[26] Notice that $\hat{F}$ has increasing hazard rate if and only if $\frac{f(q)}{1-F(q)} \frac{1}{F^{n-1}(q)+q(n-1)F^{n-2}(q)f(q)}$ increases in q. In addition, $\hat{F}$ is log-concave if and only if $\frac{f(q)}{F(q)} \frac{1}{F^{n-1}(q)+q(n-1)F^{n-2}(q)f(q)}$ decreases in q.

[27] See Sisak (2009) for a survey of other characteristics affecting the optimal allocation of prizes.

[28] See, for example, InnoCentive, IdeaConnection, and OpenIDEO.

[29] Indeed, Jeppesen and Lakhani (2010) provide evidence that the winning solution may often come from "nonobvious individuals".

For designers of innovation contests, a significant challenge is what prize to set. We model innovation contests assuming both ideas and effort are integral parts of the innovation process. When the innovation challenge is a difficult one and high-quality ideas will not be commonly observed, is it always optimal to post a high prize? We analyze this question by introducing a novel way of capturing idea scarcity with a new order of stochastic dominance.

Our analysis uncovers that while determining the prize level, contest designers should consider how much they will benefit from a marginal increase in performance as well as how difficult the challenge is. It is not necessarily the case that they should incentivize harder challenges with higher prizes. If the marginal benefit of performance is low, the optimal prize increases with the scarcity of ideas; if the marginal benefit is higher, the optimal prize decreases with the scarcity of ideas. We show that these insights continue to hold with non-linear benefit function, endogenous minimum performance, and multiple prizes.

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Appendix

A Proofs in Sections 4 and 5.1

Proof of Lemma 1. First, consider the case without the performance threshold, i.e., $t = 0$. Assume that all other solvers choose performance according to the function β_0, and assume that this function is strictly increasing and differentiable. The subscript represents the value of t. A solver's problem is

$$\max_x vF^{n-1}(\beta_0^{-1}(x)) - x/q$$

and the first order condition is

$$v\frac{dF^{n-1}(\beta_0^{-1}(x))}{dx} - \frac{1}{q} = 0$$

In equilibrium, the solver chooses performance level $x = \beta_0(q)$, so $q = \beta_0^{-1}(x)$. Denoting $\beta_0^{-1}(x)$ as y and substituting the expressions of y and q into the first order condition, we obtain

$$1 = vy\frac{dF^{n-1}(y)}{dx} \tag{A.1}$$

As a solver's idea quality $q \to 0$, his cost of any given performance level goes to infinity. Hence, the optimal performance level must converge to 0, and this yields the boundary condition $y(0) = 0$.

Note that the right hand side of (A.1) is a function of y, which means it is a differential equation with separated variables. Thus, its solution with initial condition $y(0) = 0$ is given by[30]

$$\int_0^x ds \quad = \quad v\int_0^y s\frac{dF^{n-1}(s)}{ds}ds \tag{A.2}$$

Denoting $H_0(y) = v\int_0^y s\frac{dF^{n-1}(s)}{ds}ds$, we can rewrite the above equation as $x = H_0(y) = H_0(\beta_0^{-1}(x))$, and therefore $\beta_0(x) = H_0(x)$. Thus, the performance function for every solver is

$$\beta_0(q) = v\int_0^q s\frac{dF^{n-1}(s)}{ds}ds = v\int_0^q sdF^{n-1}(s)$$

which is clearly strictly increasing and differentiable.

Assuming that all solvers other than i play according to β_0, we need to show that,

[30] See Arnold (1984), p.42 for a detailed discussion of differential equations with separated variables.

for any idea quality q of solver i, the performance $\beta_0(q)$ maximizes the expected payoff corresponding to that idea quality. Let $\pi(x,q) = vF^{n-1}(\beta_0^{-1}(x)) - x/q$ be the expected payoff of solver i with idea q that chooses performance level x. We will show that derivative $\pi_x(x,q)$ is nonnegative if x is smaller than $\beta_0(q)$ and nonpositive if x is larger than $\beta_0(q)$. As $\pi(x,q)$ is continuous in x, this implies that $x = \beta_0(q)$ maximizes $\pi(x,q)$. Notice that

$$\pi_x(x,q) = v(n-1)F^{n-2}(\beta_0^{-1}(x))f(\beta_0^{-1}(x))\frac{d\beta_0^{-1}(x)}{dx} - \frac{1}{q}$$

Let $x < \beta_0(q)$, and let $\hat{q}$ be the idea quality of a solver who is supposed to choose performance x, i.e., $\beta_0(\hat{q}) = x$. Note that $\hat{q} < q$ because β_0 is strictly increasing. Differentiating $\pi_x(x,q)$ with respect to q yields $\pi_{xq}(x,q) = 1/q^2 > 0$. Since $\hat{q} < q$, we obtain $\pi_x(x,q) \geq \pi_x(x,\hat{q})$. Since $x = \beta_0(\hat{q})$, we obtain by the first order condition that $\pi_x(x,\hat{q}) = 0$, and therefore that $\pi_x(x,q) \geq 0$ for every $x < \beta_0(q)$. A similar argument shows that $\pi_x(x,q) \leq 0$ for every $x > \beta_0(q)$.

So far we have derived the equilibrium strategy β_0 when $t = 0$. Next, we consider the case with $t > 0$ and derive the symmetric equilibrium strategy β_t. Since a solver has to choose at least t to possibly win, it is optimal for a solver with a sufficiently low idea quality to choose zero and not participate. Assume that a solver does not participate if his idea quality is below some critical level $q_t > 0$ (to be derived below). This means $\beta_t(q) = 0$ for $q < q_t$ and $\beta_t(q) \geq t$ for $q \geq q_t$. If $\beta_t(q_t) > t$, by deviating to $x = t$, a solver can achieve the same probability of winning at a lower cost. Thus, $\beta_t(q_t) = t$.

It remains to characterize $\beta_t(q)$ for $q \geq q_t$. Assume that β_t is strictly increasing and differentiable for $q \geq q_t$. Suppose all other solvers follow strategy β_t. Then, if a solver's idea quality is at the critical level q_t, he must be indifferent between choosing performance 0 and t, i.e.

$$vF^{n-1}(q_t) - t/q_t = 0$$

which can be rewritten as $q_t F^{n-1}(q_t) = t/v$ and uniquely determines q_t.

For $x \geq t$, $y = \beta_t^{-1}(x)$. Then we can verify that the first order condition is the same as (A.1). Thus, its solution y with initial condition $y(t) = q_t$ is given by

$$\int_t^x ds = v\int_t^y s\frac{dF^{n-1}(s)}{ds}ds$$

Similar to the derivation of β_0 from (A.2), the above equation implies $\beta_t(q) = t + v\int_{q_t}^q s dF^{n-1}(s)$ for $q \geq q_t$. Notice that the function β_t is equal to β_0 for $t = 0$. ■

Proof of Proposition 1. The proof has four steps.

Step 1. There exists $\bar{\lambda}_F$ such that for any $\lambda > \bar{\lambda}_F$, there is v_λ such that $\Pi'_F(v) > 0$ for $v > v_\lambda$. This means function Π_F has an increasing tail if λ is large enough.

To see this, we first show that $d \log F(q_t)/dv$ is negative. To see this, notice that (2) implies

$$\frac{dq_t}{dv} = -\frac{t}{v^2} \frac{1}{F^{n-1}(q_t) + (n-1)q_t F^{n-2}(q_t) f(q_t)}$$

so

$$\begin{aligned}
\frac{d \log F(q_t)}{dv} &= \frac{f(q_t)}{F(q_t)} \frac{dq_t}{dv} \\
&= -\frac{f(q_t)}{F(q_t)} \frac{t}{v^2} \frac{1}{F^{n-1}(q_t) + (n-1)q_t F^{n-2}(q_t) f(q_t)} \\
&= -\frac{t}{v^2} \frac{1}{n-1} \frac{1}{F^{n-1}(q_t)} \left(q_t + \frac{1}{n-1} \frac{F(q_t)}{f(q_t)} \right)^{-1} \\
&= -\frac{q_t}{v} \frac{1}{n-1} \left(q_t + \frac{1}{n-1} \frac{F(q_t)}{f(q_t)} \right)^{-1} \qquad \text{(A.3)}
\end{aligned}$$

where the last equality is from $F^{n-1}(q_t) = t/(q_t v)$.

Recall that if $\lambda = 0$, there is a unique optimal prize in $(\underline{v}_t, 1)$. Thus, if $\bar{\lambda}_F$ exists, it is larger than 0. Taking derivatives of both sides of (5) w.r.t. v, we obtain

$$\begin{aligned}
K'_F(v) &= \int_{q_t}^{w_F} \left(\int_{q_t}^{q} s dF^{n-1}(s) \right) dF^n(q) - v q_t (1 - F^n(q_t)) \frac{dF^{n-1}(q_t)}{dv} \\
&= \int_{q_t}^{w_F} \int_{s}^{w_F} s dF^n(q) dF^{n-1}(s) - v q_t (1 - F^n(q_t)) \frac{dF^{n-1}(q_t)}{dv} \\
&= \int_{q_t}^{w_F} s(1 - F^n(s)) dF^{n-1}(s) - v q_t (1 - F^n(q_t)) \frac{dF^{n-1}(q_t)}{dv} \qquad \text{(A.4)}
\end{aligned}$$

where the second equation comes from changing the order of integration. To show the existence of $\bar{\lambda}_F$, notice that (2) implies $q_t \to 0$ as $v \to +\infty$, so (A.4) implies

$$\begin{aligned}
\lim_{v \to \infty} K'_F(v) &= \int_0^{w_F} s(1 - F^n(s)) dF^{n-1}(s) \\
&\quad + \lim_{v \to \infty} \left[v q_t (1 - F^n(q_t))(n-1) F^{n-2}(q_t) \left(-\frac{dF(q_t)}{dv} \right) \right] \\
&\geq \int_0^{w_F} s(1 - F^n(s)) dF^{n-1}(s) > 0 \qquad \text{(A.5)}
\end{aligned}$$

where the first inequality comes from $dF(q_t)/dv < 0$ established above.

Next, we show $\lim_{v\to\infty} L'_F(v) = -1$. Intuitively, if the prize is large enough, every solver chooses performance above the threshold, so the marginal effect is simply the marginal cost of the prize. Formally, (4) implies

$$\begin{aligned} L'_F(v) &= F^n(q_t) - (1-v)\frac{dF^n(q_t)}{dv} - 1 \\ &= F^n(q_t)\left(1 + (1-v)n\left(-\frac{d\log F(q_t)}{dv}\right)\right) - 1 \end{aligned} \tag{A.6}$$

Substituting (A.3) into this equation, we can rewrite it as

$$L'_F(v) = F^n(q_t)\left(1 + \frac{(1-v)q_t}{v}\frac{n}{n-1}\left(q_t + \frac{1}{n-1}\frac{F(q_t)}{f(q_t)}\right)^{-1}\right) - 1$$

Rewrite (2) as $v = \frac{t}{q_t F^{n-1}(q_t)}$ and substitute it into the above expression. Then, we can rewrite $L'_F(v)$ as a function of q_t:

$$L'_F(v) = F^n(q_t) + \frac{n}{t}\frac{q_t^2 F^{2n-1}(q_t)f(q_t) - tq_t F^n(q_t)f(q_t)}{(n-1)q_t f(q_t) + F(q_t)} - 1$$

Notice that (2) implies $q_t \to 0$ as $v \to +\infty$, which means solvers with any positive idea quality participate if the prize is sufficiently high. Therefore, the above equation implies

$$\begin{aligned} \lim_{v\to\infty} L'_F(v) &= \frac{n}{t}\lim_{q_t\to 0}\frac{q_t^2 F^{2n-1}(q_t)f(q_t) - tq_t F^n(q_t)f(q_t)}{(n-1)q_t f(q_t) + F(q_t)} - 1 \\ &= -\frac{n}{t}\lim_{q_t\to 0}\frac{q_t F^n(q_t)(t - q_t F^{n-1}(q_t))}{(n-1)q_t + \frac{F(q_t)}{f(q_t)}} - 1 \\ &= -\frac{n}{t}\lim_{q_t\to 0}\frac{q_t F^n(q_t)t}{(n-1)q_t + \frac{F(q_t)}{f(q_t)}} - 1 \\ &= -n\lim_{q_t\to 0}\frac{1}{(n-1)\frac{1}{F^n(q_t)} + \frac{1}{q_t F^{n-1}(q_t)f(q_t)}} - 1 \\ &= -1 \end{aligned} \tag{A.7}$$

where the third equality is from $t > 0$ and the last from f being bounded.

Define

$$\bar{\lambda}_F = -\lim_{v\to\infty}\frac{L'_F(v)}{K'_F(v)} \tag{A.8}$$

which is in $(0, +\infty)$. Then, for $\lambda > \bar{\lambda}_F$

$$\lim_{v\to\infty} L'_F(v) + \lambda \lim_{v\to\infty} K'_F(v) \quad > \quad \lim_{v\to\infty} L'_F(v) + \bar{\lambda}_F \lim_{v\to\infty} K'_F(v) = 0$$

where the second inequality follows from (A.8). Thus, for any given $\lambda > \bar{\lambda}_F$, there is v_λ such that $\Pi'_F(v) > 0$ for $v > v_\lambda$.

Step 2. We show that $V_F(\lambda) = v_{max}$ if $\lambda > \bar{\lambda}_F$ and $V_F(\lambda) < v_{max}$ if $\lambda < \bar{\lambda}_F$. If $\lambda > \bar{\lambda}_F$, Step 1 above shows that function Π_F has an increasing tail. Then, the optimal prize is v_{max}. If $\lambda < \bar{\lambda}_F$, the definition of $\bar{\lambda}_F$ implies $V_F(\lambda) < v_{max}$.

Step 3. We show that there is a unique optimal prize if $0 \leq \lambda < \bar{\lambda}_F$. Suppose this is not true. Then, there are two prizes $v, v' \in V_F(\lambda)$ for some $\lambda \in [0, \bar{\lambda}_F)$. Step 2 above implies that v and v' are smaller than v_{max}. Then, the first order conditions are

$$L'_F(v) + \lambda K'_F(v) = 0$$
$$L'_F(v') + \lambda K'_F(v') = 0$$

Therefore, $-\frac{L'_F(v)}{K'_F(v)} = \lambda < \bar{\lambda}_F = -\lim_{v\to\infty}\frac{L'_F(v)}{K'_F(v)}$. As a result, both optimal prizes satisfy $\frac{L'_F(v)}{K'_F(v)} > \lim_{v\to\infty}\frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_F(v')}{K'_F(v')} > \lim_{v\to\infty}\frac{L'_F(v)}{K'_F(v)}$. Moreover, the objective function must have the same value at v and v', i.e.,

$$L_F(v) + \lambda K_F(v) = L_F(v') + \lambda K_F(v')$$

Solving for λ from the first order conditions and substituting into the above equation, we obtain

$$L_F(v) - \frac{L'_F(v)}{K'_F(v)}K_F(v) = L_F(v') - \frac{L'_F(v')}{K'_F(v')}K_F(v') \tag{A.9}$$

Notice that the derivative of the left hand side is $-K_F(v)\frac{d}{dv}\frac{L'_F(v)}{K'_F(v)} > 0$ due to Assumption 1. As a result, the left hand side is strictly increasing, and therefore (A.9) is impossible.

Step 4. We prove the monotonicity of $V_F(\lambda)$. From Step 2, $V_F(\lambda) = v_{max}$ for $\lambda > \bar{\lambda}_F$, so it is weakly increasing. For $\lambda < \bar{\lambda}_F$, there is a unique optimal prize because of Step 3, and it is interior because of Step 2. Thus, the optimal prize solves $\Pi'_F(v) = 0$. In addition, from their definitions, q_t decreases in v and $K_F(v)$ increases in v. Therefore, a higher λ shifts $\Pi'_F(v) = L'_F(v) + \lambda K'_F(v)$ upwards. Hence, $V_F(\lambda)$ strictly increases in λ for $\lambda < \bar{\lambda}_F$. ■

Proof of Lemma 2. The proof has two steps.

Step 1. We show that $\hat{F} \leq_{FOSD} \hat{G}$ implies $F \leq_{FOSD} G$. We first show that $F(0) \geq G(0)$. Suppose otherwise that $F(0) < G(0)$. Notice that $\lim_{x\to 0}\hat{F}(x) = \lim_{x\to 0}F(\phi_F^{-1}(x)) = F(0)$. Similarly, $\lim_{x\to 0}\hat{G}(x) = G(0)$. Thus, $F(0) < G(0)$ implies $\lim_{x\to 0}\hat{F}(x) < \lim_{x\to 0}\hat{G}(x)$, which contradicts $\hat{F} \leq_{FOSD} \hat{G}$.

Next, we prove that $F \leq_{FOSD} G$. Suppose otherwise that there exists $q' \in [0, w_F]$ such that $F(q') < G(q')$. Then, Step 1 and the intermediate value theorem imply that there is $q \in (0, w_F)$ such that $F(q) = G(q)$. Define $\hat{q} = \max\{q \in (0, q') | F(q) = G(q)\}$. Then, $F(\hat{q}) = G(\hat{q})$ and $F(\hat{q} + \varepsilon) < G(\hat{q} + \varepsilon)$ for sufficiently small $\varepsilon > 0$. Moreover, let $\hat{x} = \hat{q}F^{n-1}(\hat{q}) = \hat{q}G^{n-1}(\hat{q})$. Then, by their definitions, $\phi_F(\hat{q} + \varepsilon) < \phi_G(\hat{q} + \varepsilon)$ and $\phi_F^{-1}(\hat{x}+\Delta) > \phi_G^{-1}(\hat{x}+\Delta)$ for sufficiently small $\Delta > 0$. Recall that $\phi_F^{-1}(x)F^{n-1}(\phi_F^{-1}(x)) = x$ and $\phi_G^{-1}(x)G^{n-1}(\phi_G^{-1}(x)) = x$, so

$$\phi_F^{-1}(x)F^{n-1}(\phi_F^{-1}(x)) = \phi_G^{-1}(x)G^{n-1}(\phi_G^{-1}(x)).$$

Since $\phi_F^{-1}(\hat{x} + \Delta) > \phi_G^{-1}(\hat{x} + \Delta)$, the above equation implies $F^{n-1}(\phi_F^{-1}(\hat{x} + \Delta)) < G^{n-1}(\phi_F^{-1}(\hat{x} + \Delta))$, which is equivalent to $\hat{F}(\hat{x} + \Delta) < \hat{G}(\hat{x} + \Delta)$. This contradicts $\hat{F} \leq_{FOSD} \hat{G}$.

Step 2. We show that $F \leq_{FOSD} G$ implies $\hat{F} \leq_{FOSD} \hat{G}$. Because $F \leq_{FOSD} G$, we have $qF^{n-1}(q) \geq qG^{n-1}(q)$. Then the solution of

$$qF^{n-1}(q) = x \tag{A.10}$$

must be smaller than that of

$$qG^{n-1}(q) = x \tag{A.11}$$

That is, $\phi_F^{-1}(x) \leq \phi_G^{-1}(x)$. Notice that (A.10) and (A.11) imply

$$\phi_F^{-1}(x)F^{n-1}(\phi_F^{-1}(x)) = \phi_G^{-1}(x)G^{n-1}(\phi_G^{-1}(x))$$

so $\phi_F^{-1}(x) \leq \phi_G^{-1}(x)$ implies $F^{n-1}(\phi_F^{-1}(x)) \geq G^{n-1}(\phi_G^{-1}(x))$, which combined with the definitions of $\hat{F}$ and $\hat{G}$ implies $\hat{F}(x) \geq \hat{G}(x)$. ■

Proof of Lemma 3. Using the definition of $\hat{F}$, we have

$$\frac{d \log \hat{F}(x)}{dx} = \frac{d \log F(\phi_F^{-1}(x))}{dx} = \frac{f(\phi_F^{-1}(x))}{F(\phi_F^{-1}(x))} \frac{d\phi_F^{-1}(x)}{dx}$$

Denote $q_x \equiv \phi_F^{-1}(x)$ and substitute it into the equation above. We obtain

$$\begin{aligned}\frac{d\log \hat{F}(x)}{dx} &= \frac{f(q_x)}{F(q_x)}\frac{1}{\phi_F'(q_x)} \\ &= \frac{f(q_x)}{F(q_x)}\frac{1}{F^{n-1}(q_x)+(n-1)q_xF^{n-2}(q_x)f(q_x)} \\ &= \frac{1}{n-1}\frac{1}{F^{n-1}(q_x)}\left(q_x+\frac{1}{n-1}\frac{F(q_x)}{f(q_x)}\right)^{-1} \end{aligned} \tag{A.12}$$

Notice that if $q+F(q)/f(q)$ is non-decreasing, $q_x+\frac{1}{n-1}\frac{F(q_x)}{f(q_x)}$ is increasing in q_x for $n \geq 2$. Moreover, if x increases, q_x increases. Therefore, (A.12) decreases in x, which means $\hat{F}$ is log-concave. ■

Proof of Lemma 4. It is well-known that $\hat{F} \prec_{LR} \hat{G}$ implies $\hat{F} \prec_{FOSD} \hat{G}$, which, combined with Lemma 2, implies $F \prec_{FOSD} G$. ■

B Proofs of Propositions 2 and 3

We first present Lemmas A.1, A.2 and A.3 which we use to prove Proposition 2.

Lemma A.1 *Suppose $F \prec G$. Then, $L_F'(v) = 0$ implies $L_G'(v) < 0$.*

Proof. Using equation (A.6), we can rewrite $L_F'(v) = 0$ as

$$F^n(q_t) - (1-v)\frac{dF^n(q_t)}{dv} - 1 = 0 \tag{A.13}$$

Similarly, we can rewrite $L_G'(v) < 0$ as

$$G^n(q_t') - (1-v)\frac{dG^n(q_t')}{dv} - 1 < 0 \tag{A.14}$$

Suppose $L_F'(v) = 0$. Then equation (A.13) implies

$$1 - v = (F^n(q_t) - 1)\left(\frac{dF^n(q_t)}{dv}\right)^{-1}$$

Substituting it into (A.14), we can rewrite $L_G'(v) < 0$ as

$$G^n(q_t') - (F^n(q_t) - 1)\left(\frac{dF^n(q_t)}{dv}\right)^{-1}\frac{dG^n(q_t')}{dv} - 1 < 0 \tag{A.15}$$

It remains to show (A.15). Recall that in the first step to prove Proposition 1, we obtain $dF^n(q_t)/dv < 0$ and $dG^n(q_t')/dv < 0$, so (A.15) can be rewritten as

$$-\frac{dF^n(q_t)}{dv}\frac{1}{1-F^n(q_t)} > -\frac{dG^n(q_t')}{dv}\frac{1}{1-G^n(q_t')} \tag{A.16}$$

Recall that the definition of $F \prec G$ requires $\hat{F} \prec_{LR} \hat{G}$, which implies $\hat{F}^n \prec_{LR} \hat{G}^n$ due to Theorem 1.C.33 of Shaked and Shanthikumar (2007). Then, (A.16) holds because

$$LHS \text{ of (A.16)} = \frac{(\hat{F}^n)'(t/v)}{1-\hat{F}^n(t/v)}\frac{t}{v^2} > \frac{(\hat{G}^n)'(t/v)}{1-\hat{G}^n(t/v)}\frac{t}{v^2} = RHS \text{ of (A.16)}$$

where the equalities follow from the definitions of $\hat{F}$ and $\hat{G}$ and the inequality follows from the hazard rate dominance implied by $\hat{F}^n \prec_{LR} \hat{G}^n$. Hence, (A.15) also holds. ■

Lemma A.2 $K_F'(v) > 0$. *Moreover,* $F \prec G$ *implies* $K_F'(v) < K_G'(v)$.

Proof. If v increases, q_t decreases. Therefore, (5) implies that $K_F(v)$ is increasing in v.

We prove $K_F'(v) < K_G'(v)$ in two steps. First, we prove

$$q_t(1-F^n(q_t))\frac{dF^{n-1}(q_t)}{dv} \geq q_t'(1-G^n(q_t'))\frac{dG^{n-1}(q_t')}{dv} \tag{A.17}$$

Notice that equation (2) implies $q_t = \frac{t}{v}\frac{1}{F^{n-1}(q_t)}$. Substituting this expression into (A.17), we have

$$\begin{aligned} LHS \text{ of (A.17)} &= \frac{t}{v}(1-F^n(q_t))\frac{dF^{n-1}(q_t)}{dv}\frac{1}{F^{n-1}(q_t)} \\ &= \frac{t}{v}(1-F^n(q_t))\frac{d\hat{F}^{\frac{n-1}{n}}(t/v)}{d(t/v)}\frac{1}{\hat{F}^{\frac{n-1}{n}}(t/v)}\left(-\frac{t}{v^2}\right) \end{aligned} \tag{A.18}$$

where the second equality is from $\hat{F}(t/v) = F(q_t)$.

Notice that $\hat{F} \prec_{LR} \hat{G}$ implies $\hat{G}$ dominates $\hat{F}$ in terms of reverse hazard rate. Therefore, $\hat{G}^{\frac{n-1}{n}}$ also dominates $\hat{F}^{\frac{n-1}{n}}$ in terms of reverse hazard rate. That is,

$$\frac{d\hat{F}^{\frac{n-1}{n}}(t/v)}{d(t/v)}\frac{1}{\hat{F}^{\frac{n-1}{n}}(t/v)} \leq \frac{d\hat{G}^{\frac{n-1}{n}}(t/v)}{d(t/v)}\frac{1}{\hat{G}^{\frac{n-1}{n}}(t/v)}$$

As a result, equation (A.18) implies

$$LHS \text{ of (A.17)} \geq \frac{t}{v}(1-F^n(q_t))\frac{d\hat{G}^{\frac{n-1}{n}}(t/v)}{d(t/v)}\frac{1}{\hat{G}^{\frac{n-1}{n}}(t/v)}\left(-\frac{t}{v^2}\right) \geq RHS \text{ of (A.17)}$$

Second, we use equation (A.17) to prove Lemma A.2. Following the same argument as for (A.4), we have

$$K'_G(v) = \int_{q'_t}^{w_G} s(1 - G^n(s))dG^{n-1}(s) - vq'_t(1 - G^n(q'_t))\frac{dG^{n-1}(q'_t)}{dv}$$

Because of (A.17), it is sufficient to show

$$\int_{q_t}^{w_F} s(1 - F^n(s))dF^{n-1}(s) \leq \int_{q'_t}^{w_G} s(1 - G^n(s))dG^{n-1}(s)$$

Changing variables using $F(s) = x$ and $s = F^{-1}(x)$ on the left hand side and $G(s) = x$ and $s = G^{-1}(x)$ on the right hand side, we can rewrite the above inequality as

$$\int_{F(q_t)}^{1} F^{-1}(x)(1 - x^n)dx^{n-1} \leq \int_{G(q'_t)}^{1} G^{-1}(x)(1 - x^n)dx^{n-1} \tag{A.19}$$

To see (A.19), recall that $G(q) \leq F(q)$ according to Lemma 2. Therefore, if we replace F in (2) with G, the left hand side shifts down, so $q'_t \geq q_t$. In addition, notice that $q_t F^{n-1}(q_t) = q'_t G^{n-1}(q'_t)$, so $q'_t \geq q_t$ implies that $F(q_t) \geq G(q'_t)$. Moreover, $G(q) \leq F(q)$ implies $F^{-1}(x) \leq G^{-1}(x)$. Hence, the integrand on the left hand side of (A.19) is smaller than that on the right hand side, and the integration on the left is over a smaller set because $F(q_t) \geq G(q'_t)$, so (A.19) holds. ■

According to Proposition 1, the optimal prize reaches the upper boundary v_{max} if and only if $\lambda \geq \bar{\lambda}_F$, with $\bar{\lambda}_F$ defined in (A.8). If the distribution is G instead of F, we can define $\bar{\lambda}_G = -\lim_{v\to\infty} \frac{L'_G(v)}{K'_G(v)}$ analogously, and the optimal prize reaches v_{max} if and only if $\lambda \geq \bar{\lambda}_G$.

Lemma A.3 *If $F \prec G$, then $0 < \bar{\lambda}_G < \bar{\lambda}_F < +\infty$.*

Proof. We prove in two steps. First, we show $\lim_{v\to\infty} K'_F(v) < \lim_{v\to\infty} K'_G(v)$. Since $\lim_{v\to\infty} q_t = \lim_{v\to\infty} q'_t = 0$, as in the proof of Lemma A.2 it is sufficient to show

$$\int_{0}^{w_F} s(1 - F^n(s))dF^{n-1}(s) < \int_{0}^{w_G} s(1 - G^n(s))dG^{n-1}(s) \tag{A.20}$$

Because $G(q) \leq F(q)$ and the inequality holds strictly for a set of positive measure, we can use this inequality to prove (A.20) in the same way we prove (A.19).

Second, we prove the lemma. In the proof of Proposition 1, we show $\bar{\lambda}_F \in (0, +\infty)$, so similarly, $\bar{\lambda}_G \in (0, +\infty)$. To see $\bar{\lambda}_G < \bar{\lambda}_F$, notice

$$\bar{\lambda}_G = -\lim_{v\to\infty} \frac{L'_G(v)}{K'_G(v)} = \frac{1}{\lim\limits_{v\to\infty} K'_G(v)} < \frac{1}{\lim\limits_{v\to\infty} K'_F(v)} = -\lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)} = \bar{\lambda}_F$$

where the first and last equalities are from the definitions of $\bar{\lambda}_G$ and $\bar{\lambda}_F$, the second and third equalities are from (A.7), and the inequality is from the first step. Hence, $0 < \bar{\lambda}_G < \bar{\lambda}_F < +\infty$. ■

Proof of Proposition 2. We proceed in three steps.

Step I. $V_F(\lambda) > V_G(\lambda)$ for λ close to 0. To see this, recall that if $\lambda = 0$, the expected profit is $L_F(v) = (1 - v)(1 - F^n(q_t))$, which is zero or negative for $v = 0$ or $v \geq 1$. Thus, its maximum is reached at an interior prize value. Therefore, both $V_F(0)$ and $V_G(0)$ satisfy the first order conditions $L'_F(V_F(0)) = 0$ and $L'_G(V_G(0)) = 0$, which combined with Lemma A.1 imply that $V_G(0) < V_F(0)$. By continuity of $\Pi_F(v)$ in λ, if λ is sufficiently close to 0, we still have $V_G(\lambda) < V_F(\lambda)$.

Step II. $V_F(\lambda) \leq V_G(\lambda)$ if λ is large enough. Lemma A.3 implies that $V_F(\lambda) \leq V_G(\lambda) = v_{max}$ for $\lambda > \bar{\lambda}_G$.

Step III. In this step, we show existence and uniqueness of $\hat{\lambda}$. From Step I, $V_F(\lambda) > V_G(\lambda)$ for λ close to 0. Consider three cases:

Case 1. $V_F(\lambda) > V_G(\lambda)$ for all $\lambda \leq \bar{\lambda}_G$. Then, $\hat{\lambda} = \bar{\lambda}_G$ and the proposition holds.

Case 2. $V_F(\lambda) = V_G(\lambda)$ for some $\lambda \in (0, \bar{\lambda}_G]$. Notice that for any λ such that $V_F(\lambda) = V_G(\lambda) = v$, the first order conditions are

$$L'_F(v) + \lambda K'_F(v) = 0 \tag{A.21}$$

$$L'_G(v) + \lambda K'_G(v) = 0 \tag{A.22}$$

Since $\lambda \in (0, \bar{\lambda}_G]$, we have $\lambda < \bar{\lambda}_F$, so v is an interior maximizer. As in Step 3 of Proposition 1, the interior optimal prize v satisfies $\frac{L'_F(v)}{K'_F(v)} > \lim\limits_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_G(v)}{K'_G(v)} > \lim\limits_{v\to\infty} \frac{L'_G(v)}{K'_G(v)}$. Solving (A.21) for λ and substituting into (A.22) yields

$$L'_F(v)/K'_F(v) = L'_G(v)/K'_G(v)$$

This equation has a unique solution because $L'_F(v)/K'_F(v)$ and $L'_G(v)/K'_G(v)$ cross at most once over v values such that $\frac{L'_F(v)}{K'_F(v)} > \lim\limits_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_G(v)}{K'_G(v)} > \lim\limits_{v\to\infty} \frac{L'_G(v)}{K'_G(v)}$ due to Assumption 2. Thus, there is a unique λ such that $V_F(\lambda) = V_G(\lambda)$. Letting $\hat{\lambda}$ be this unique λ gives us the proposition.

Case 3. $V_F(\lambda) < V_G(\lambda)$ for some $\lambda_0 \in (0, \bar{\lambda}_G]$. Notice that if V_F is discontinuous at an interior point of $(0, \bar{\lambda}_F)$, there are at least two optimal prizes at this interior λ. This contradicts with Proposition 1, so V_F is continuous over $(0, \bar{\lambda}_F)$. Similarly, V_G is continuous over $(0, \bar{\lambda}_G)$. Recall that $V_F(\lambda_0) < V_G(\lambda_0)$, so $\lambda_0 < \bar{\lambda}_G < \bar{\lambda}_F$ and therefore V_F and V_G are continuous over $(0, \lambda_0)$. Moreover, $V_F(\lambda) > V_G(\lambda)$ if λ is sufficiently close to 0 and that $V_F(\lambda_0) < V_G(\lambda_0)$, so the intermediate value theorem implies that there is a $\lambda \in (0, \lambda_0)$ such that $V_F(\lambda) = V_G(\lambda)$. Then, we can prove the proposition as in Case 2. ■

Proof of Proposition 3. If $\lambda = 0$, the seeker's expected profit is $L_F(v) = (1 - v)(1 - F^n(q_t))$. If $v \leq t/w_F$, $F(q_t) = 1$ and $L_F(v) = 0$. If $v \geq 1$, we also have $L_F(v) \leq 0$. Thus, if $\lambda = 0$, the optimal prizes are interior and satisfy the first order condition (A.13). Following the rest of Lemma A.1's proof, we can verify that Lemma A.1 remains true without Assumptions 1 and 2. Then, Theorem 4 of Milgrom and Shannon (1994) implies $V_F(\lambda) \geq V_G(\lambda)$. It is straightforward to see that Lemma A.1 remains true for λ sufficiently close to 0, and therefore $V_F(\lambda) \geq V_G(\lambda)$ for sufficiently small λ.

Notice that the proofs of Lemmas A.2 and A.3 do not rely on Assumptions 1 and 2. Thus, the lemmas imply for sufficient large λ, the optimal prize is v_{max} and therefore $V_G(\lambda) \geq V_F(\lambda)$. ■

Online Appendix

C Parametric Distributions Illustrating Assumptions 1 and 2

We show that Assumptions 1 and 2 are satisfied by many distribution families. We first consider uniform and power function distributions, for which we have analytical results. We then present numerical results for other distribution families.

C.1 Assumption 1 for Uniform and Power Function Distributions

Notice that $\frac{L'_F(v)}{K'_F(v)}$ is decreasing if and only if $L_F(v) - \frac{L'_F(v)}{K'_F(v)} K_F(v)$ is increasing, because the derivative of $L_F(v) - \frac{L'_F(v)}{K'_F(v)} K_F(v)$ is $-\frac{d}{dv}\left(\frac{L'_F(v)}{K'_F(v)}\right) K_F(v)$. Using $x = t/v$ we can rewrite

$$\begin{aligned} & L_F(v) - \frac{L'_F(v)}{K'_F(v)} K_F(v) \\ = \; & \left(1 - \frac{t}{x}\right)(1 - \hat{F}^n(x)) + \frac{\frac{t}{x^2}(1 - \hat{F}^n(x)) + \left(1 - \frac{t}{x}\right)(1 - \hat{F}^n(x))'}{\frac{1}{x} + \frac{x(1 - \hat{F}^n(x))\frac{\hat{f}(x)}{\hat{F}(x)}}{\int_x^{w_F} y(1 - \hat{F}^n(y))\frac{\hat{f}(y)}{\hat{F}(y)} dy}} \end{aligned} \tag{A.23}$$

Thus, to verify Assumption 1, it suffices to show (A.23) is strictly decreasing in $x \in (0, w_F)$.

- **Uniform Distributions** Consider $F(q) = q/w_F$. Then, $\phi_F(q) = q(q/w_F)^{n-1}$, so $\phi_F^{-1}(x) = (w_F)^{\frac{n-1}{n}} x^{\frac{1}{n}}$ and $\hat{F}(x) = (x/w_F)^{1/n}$. Substituting these expressions into (A.23), we have

$$L_F(v) - L'_F(v) K_F(v)/K'_F(v) = \frac{1 - \frac{x}{w_F}}{1 + \frac{x}{w_F}} \left(1 - \frac{t}{w_F}\right)$$

  which is decreasing in x since $t < w_F$. Thus, $F(q) = q/w_F$ satisfies Assumption 1.

- **Power Function Distributions** Consider $F(q) = q^\alpha$ with $q \in [0, 1]$ and $0 < \alpha$. Then, $\phi_F(q) = q^{(n-1)\alpha+1}$, so $\phi_F^{-1}(x) = x^{\frac{1}{(n-1)\alpha+1}}$ and $\hat{F}(x) = x^{\frac{\alpha}{(n-1)\alpha+1}}$. The following lemma verifies Assumption 1 for power function distributions.

Lemma A.4 *If $F(q) = q^\alpha$ with $\alpha > 0$, Assumption 1 is satisfied.*

Proof. Recall that Assumption 1 requires $\frac{L'_F(v)}{K'_F(v)}$ to be strictly decreasing in v whenever $\frac{L'_F(v)}{K'_F(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$.

First, we rewrite $\frac{L'_F(v)}{K'_F(v)}$ as a function of x. With power function distributions, we can rewrite

$$\begin{aligned}
L_F(v) &= \left(1-\frac{t}{x}\right)\left(1-x^{\frac{n\alpha}{(n-1)\alpha+1}}\right) = \left(1-\frac{t}{x}\right)\left(1-x^A\right) \\
\frac{dL_F(v)}{dx} &= \frac{t}{x^2}\left(1-x^{\frac{n\alpha}{(n-1)\alpha+1}}\right) + \left(1-\frac{t}{x}\right)(-1)\frac{n\alpha}{(n-1)\alpha+1}x^{\frac{n\alpha}{(n-1)\alpha+1}-1} \\
&= \frac{t}{x^2}\left(1-x^A\right) - \left(1-\frac{t}{x}\right)Ax^{A-1} \qquad \text{(A.24)}
\end{aligned}$$

where

$$A = \frac{n\alpha}{(n-1)\alpha+1}$$

In addition,

$$\begin{aligned}
x(1-\hat{F}^n(x))\frac{\hat{f}(x)}{\hat{F}(x)} &= x(1-x^{\frac{n\alpha}{(n-1)\alpha+1}})\frac{\frac{\alpha}{(n-1)\alpha+1}x^{\frac{\alpha}{(n-1)\alpha+1}-1}}{x^{\frac{\alpha}{(n-1)\alpha+1}}} \\
&= \frac{\alpha}{(n-1)\alpha+1}(1-x^{\frac{n\alpha}{(n-1)\alpha+1}})
\end{aligned}$$

so

$$\begin{aligned}
&\int_x^1 y(1-\hat{F}^n(y))\frac{\hat{f}(y)}{\hat{F}(y)}dy \\
=& \int_x^1 \frac{\alpha}{(n-1)\alpha+1}(1-y^{\frac{n\alpha}{(n-1)\alpha+1}})dy \\
=& \frac{\alpha}{(n-1)\alpha+1}(1-x) - \frac{\alpha}{(n-1)\alpha+1}\frac{1}{\frac{n\alpha}{(n-1)\alpha+1}+1}(1-x^{\frac{n\alpha}{(n-1)\alpha+1}+1}) \\
=& \frac{1}{n}\left(A(1-x) - A\frac{1}{A+1}(1-x^{A+1})\right)
\end{aligned}$$

Using the above expressions, we have

$$\begin{aligned}
K_F(v) &= (n-1)\frac{t}{x}\int_x^1 (1-\hat{F}(y))y\frac{\hat{f}(y)}{\hat{F}(y)}dy \\
&= A\frac{n-1}{n}\frac{t}{x}\left((1-x) - \frac{1}{A+1}(1-x^{A+1})\right)
\end{aligned}$$

and

$$\begin{aligned}\frac{dK_F(v)}{dx} &= A\frac{n-1}{n}\left[-\frac{t}{x^2}\left((1-x)-\frac{1}{A+1}(1-x^{A+1})\right)+\frac{t}{x}\left(-1+x^A\right)\right]\\ &= -A\frac{n-1}{n}\frac{t}{x}\left(1-\frac{1}{A+1}\right)\left(\frac{1}{x}-x^A\right)\end{aligned} \tag{A.25}$$

Equations (A.24) and (A.25) imply

$$\begin{aligned}\frac{L_F'(v)}{K_F'(v)} &= \frac{\frac{t}{x^2}(1-x^A)-\left(1-\frac{t}{x}\right)Ax^{A-1}}{-A\frac{n-1}{n}\frac{t}{x}\left(1-\frac{1}{A+1}\right)\left(\frac{1}{x}-x^A\right)}\\ &= -\frac{n}{n-1}\frac{1}{t}\frac{1}{A\left(1-\frac{1}{A+1}\right)}\frac{t-x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}}\end{aligned} \tag{A.26}$$

Next, consider the inequality $\frac{L_F'(v)}{K_F'(v)} > \lim_{v\to\infty}\frac{L_F'(v)}{K_F'(v)}$. Using (A.26), we have

$$\begin{aligned}\lim_{v\to\infty}\frac{L_F'(v)}{K_F'(v)} &= \lim_{x\to 0}\left(-\frac{n}{n-1}\frac{1}{t}\frac{1}{A\left(1-\frac{1}{A+1}\right)}\frac{t-x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}}\right)\\ &= -\frac{n}{n-1}\frac{1}{A\left(1-\frac{1}{A+1}\right)}\end{aligned}$$

Combining the above expression with (A.26), we can rewrite $\frac{L_F'(v)}{K_F'(v)} > \lim_{v\to\infty}\frac{L_F'(v)}{K_F'(v)}$ as

$$-\frac{n}{n-1}\frac{1}{t}\frac{1}{A\left(1-\frac{1}{A+1}\right)}\frac{t-x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}} > -\frac{n}{n-1}\frac{1}{A\left(1-\frac{1}{A+1}\right)}$$

which can be simplified to

$$(A-t)x > (A-1)t \tag{A.27}$$

Thus, for Assumption 1 it is sufficient to show $\frac{L_F'(v)}{K_F'(v)}$ is strictly increasing in x whenever (A.27) holds. According to (A.26), for $\frac{L_F'(v)}{K_F'(v)}$ to be strictly increasing in x, we need $\frac{t-x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}}$ strictly decreasing in x, which requires its derivative w.r.t. x to be negative:

$$\frac{\left(-(1-A)tAx^{A-1}-A(1+A)x^A\right)(1-x^{1+A})+(t-(1-A)tx^A-Ax^{1+A})(1+A)x^A}{(1-x^{1+A})^2} < 0$$

Collecting terms, we can simplify the above inequality to

$$(1-A)tA+(A-t)(1+A)x+(1-A)tx^{1+A} > 0 \tag{A.28}$$

Now for Assumption 1, we need to show that (A.27) implies (A.28). We prove in two cases.

Case 1. Suppose $A \leq 1$. Notice the left hand side of (A.27) is linear in x, and it holds for $x = 0$ and $x = 1$. Therefore, (A.27) holds for all $x \in (0, 1)$. As a result, we need to show (A.28) for all x.

Notice that the left hand side of (A.28) is linear in t, so it suffices to show it for $t = 0$ and $t = 1$. If $t = 0$, (A.28) reduces to $A(1 + A)x > 0$, which is true. If $t = 1$, the left hand side of (A.28) becomes

$$(1 - A)A + (A - 1)(1 + A)x + (1 - A)x^{1+A}$$

and its derivative w.r.t. x is

$$(A - 1)(1 + A) + (1 - A)(1 + A)x^{A}$$

Since $A \leq 1$, it is straightforward to verify that the derivative is negative for $x \in (0, 1)$ and equals to 0 at $x = 1$. This means the left hand side of (A.28) is strictly decreasing in x, its minimum is reached at $x = 1$, and the minimum is $(1-A)A+(A-1)(1+A)+(1-A) = 0$. Therefore, (A.28) holds for $t = 1$ as well. Since it holds for both $t = 0$ and $t = 1$, we have (A.27) implies (A.28) in this case.

Case 2. Suppose $A > 1$. Then, (A.27) implies

$$x > \frac{(A - 1)t}{A - t} \tag{A.29}$$

In addition,

$$\begin{aligned}
\text{LHS of (A.28)} &= (1 - A)tA + \underbrace{(A - t)(1 + A)}_{>0} x + (1 - A)tx^{1+A} \\
&> (1 - A)tA + (A - t)(1 + A)\frac{(A - 1)t}{A - t} + (1 - A)tx^{1+A} \\
&= (1 - A)tA + (A - 1)t + A(A - 1)t + (1 - A)tx^{1+A} \\
&= (A - 1)t(1 - x^{1+A}) > 0
\end{aligned}$$

where the inequality is from (A.29). Thus, (A.28) holds.

Combining Cases 1 and 2, we verify that (A.27) implies (A.28), and therefore Assumption 1 holds. ■

C.2 Assumption 2 for Uniform and Power Function Distributions

We can rewrite

$$\frac{L'_F(v)}{K'_F(v)} = \frac{\left[\left(1-\frac{t}{x}\right)\left(1-\hat{F}^n(x)\right)\right]'}{\left[\frac{t}{x}\int_x^{w_F}(1-\hat{F}^n(y))(n-1)\frac{y}{\hat{F}(y)}\hat{f}(y)dy\right]'} \tag{A.30}$$

$$\frac{L'_G(v)}{K'_G(v)} = \frac{\left[\left(1-\frac{t}{x}\right)\left(1-\hat{G}^n(x)\right)\right]'}{\left[\frac{t}{x}\int_x^{w_G}(1-\hat{G}^n(y))(n-1)\frac{y}{\hat{G}(y)}\hat{g}(y)dy\right]'} \tag{A.31}$$

Thus, to verify Assumption 2, it suffices to show (A.30) and (A.31) cross at most once for $x \in (0, w_F)$.

- **Uniform Distributions** Consider $F(q) = q/w_F$ and $G(q) = q/w_G$ with $w_G > w_F > 0$. Then, (A.30) and (A.31) become

$$\frac{L'_F(v)}{K'_F(v)} = -\frac{n}{n-1}\frac{2}{t}\frac{tw_F - x^2}{w_F^2 - x^2}$$
$$\frac{L'_G(v)}{K'_G(v)} = -\frac{n}{n-1}\frac{2}{t}\frac{tw_G - x^2}{w_G^2 - x^2}$$

  Suppose the two terms cross at x, then x satisfies

$$\frac{tw_F - x^2}{w_F^2 - x^2} = \frac{tw_G - x^2}{w_G^2 - x^2}$$

  which has a unique solution $x = \sqrt{\frac{tw_Fw_G}{w_G+w_F-t}}$. Thus, Assumption 2 is satisfied.

- **Power Function Distributions** Consider $F(q) = q^\alpha$. Then, $\phi_F(q) = q^{(n-1)\alpha+1}$, so $\phi_F^{-1}(x) = x^{\frac{1}{(n-1)\alpha+1}}$ and $\hat{F}(x) = x^{\frac{\alpha}{(n-1)\alpha+1}}$. Then, (A.30) can be rewritten as

$$\frac{L'_F(v)}{K'_F(v)} = -\frac{n}{n-1}\frac{1}{t}\frac{1}{A\left(1-\frac{1}{A+1}\right)}\frac{t - x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}}$$

  where $A = \frac{n\alpha}{(n-1)\alpha+1}$. Similarly,

$$\frac{L'_G(v)}{K'_G(v)} = -\frac{n}{n-1}\frac{1}{t}\frac{1}{A'\left(1-\frac{1}{A'+1}\right)}\frac{t - x^{A'+1}\left((1-A')\frac{t}{x}+A'\right)}{1-x^{A'+1}}$$

  where $A' = \frac{n\alpha'}{(n-1)\alpha'+1}$ and $A' > A$.

Therefore, to show that $\frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_G(v)}{K'_G(v)}$ cross at most once, it is sufficient to show that

$$\frac{\frac{t-x^{A'+1}\left((1-A')\frac{t}{x}+A'\right)}{1-x^{A'+1}}}{\frac{t-x^{A+1}\left((1-A)\frac{t}{x}+A\right)}{1-x^{A+1}}} = \frac{\frac{1+A}{A^2}}{\frac{1+A'}{A'^2}} \tag{A.32}$$

has at most one solution $x \in (0, 1)$. Next, we prove this statement in a number of steps, which are orgaized as a sequence of claims and the final step is in Lemma A.5.

Claim A.1

$$\frac{1-x^A}{A}\frac{1-x^{1+A}}{(1+A)(1-x)-(1-x^{1+A})^2} \leq -\frac{\ln x}{x} \tag{A.33}$$

Proof. There is a basic inequality of logarithmic function[31]

$$\ln x \leq -\frac{2(1-x)}{1+x}$$

for $x \in (0, 1)$. Replacing x by x^A, we obtain

$$\ln x \leq -\frac{2(1-x^A)}{(1+x^A)A}$$

Therefore, to show (A.33), it is sufficient to prove

$$\frac{1-x^A}{A}\frac{1-x^{1+A}}{(1+A)(1-x)-(1-x^{1+A})^2} \leq \frac{2(1-x^A)}{(1+x^A)Ax}$$

or equivalently

$$\frac{1-x^{1+A}}{(1+A)(1-x)-(1-x^{1+A})^2} \leq \frac{2}{(1+x^A)x} \tag{A.34}$$

Next, we verify the denominator on the LHS is positive: $(1+A)(1-x)-(1-x^{1+A})^2 > 0$. Its derivative w.r.t. x is

$$-(1+A)-2(1-x^{1+A})(-(1+A))x^A \tag{A.35}$$

[31] See for instance Topsøe, F. (2004). “Some Bounds for the Logarithmic Function,” *RGMIA Res. Rep. Collection*, 7(2), pp. 1-20.

which has the same sign as

$$-1 + 2(1 - x^{1+A})x^A \tag{A.36}$$

Let us consider the maximum of (A.36) as a function of $A \in [0, 2]$. If the maximum is reached at an interior point $A \in (0, 2)$, it must satisfy the first order condition:

$$-2x^{1+A}x^A \ln x + 2(1 - x^{1+A})x^A \ln x = 0$$

which implies $x^{1+A} = 1/2$. Thus, at an interior maximizer, (A.36)$= -1 + 2(1 - \frac{1}{2})x^A = -1 + x^A < 0$, where the equality is from $x^{1+A} = 1/2$. At the lower boundary $A = 0$, (A.36) $= -1 < 0$. At the upper boundary $A = 2$, (A.36) $= -1 + 2(1 - x^3)x^2 < 0$. Thus, (A.35) and (A.36) are both negative.

This means the denominator $(1 + A)(1 - x) - (1 - x^{1+A})^2$ is decreasing in x, so

$$(1 + A)(1 - x) - (1 - x^{1+A})^2 \geq (1 + A)(1 - 0) - 1 = A > 0$$

Now we know the denominators in (A.34) are positive, we can rewrite it as

$$\begin{aligned} (1 - x^{1+A})(x + x^{1+A}) &< 2(1 + A)(1 - x) - (1 - x^{1+A})^2 \\ x - x^{2+A} &< 2(1 + A)(1 - x) - 1 + x^{1+A} \\ 2(1 + A)(1 - x) - (1 + x)(1 - x^{1+A}) &> 0 \end{aligned}$$

Notice the LHS is larger than

$$2(1 + A)(1 - x) - 2(1 - x^{1+A}) \tag{A.37}$$

so it is sufficient to show (A.37) is positive. To see this, taking the derivative of (A.37) w.r.t. x, we obtain

$$-2(1 + A) + 2(1 + A)x^A = -2(1 + A)(1 - x^A) \leq 0$$

This means (A.37) is decreasing in x, so (A.37) $\geq 2(1+A)(1-1)-2(1-1) = 0$. Therefore, (A.34) is true and so is (A.33). ■

Claim A.2 $\left(\frac{1}{A}x^{1+A} + 1\right)\dfrac{1 - x^A}{1 - x^{1+A}}$ *increases in* $A \in (0, 2)$.

Proof. Differentiating the logarithm of the above expression, we obtain

$$\begin{aligned}
& \frac{d}{dA}\left(\ln\left(\frac{1}{A}x^{1+A}+1\right)+\ln(1-x^A)-\ln(1-x^{1+A})\right) \\
= & \frac{-\frac{1}{A^2}x^{1+A}+\frac{1}{A}x^{1+A}\ln x}{\frac{1}{A}x^{1+A}+1}-\frac{x^A\ln x}{1-x^A}+\frac{x^{1+A}\ln x}{1-x^{1+A}} \\
= & \frac{-\frac{1}{A^2}x^{1+A}}{\frac{1}{A}x^{1+A}+1}+\ln x\left(\frac{\frac{1}{A}x^{1+A}}{\frac{1}{A}x^{1+A}+1}-\frac{x^A}{1-x^A}+\frac{x^{1+A}}{1-x^{1+A}}\right) \qquad \text{(A.38)}
\end{aligned}$$

Next, we show

$$\frac{\frac{1}{A}x^{1+A}}{\frac{1}{A}x^{1+A}+1}-\frac{x^A}{1-x^A}+\frac{x^{1+A}}{1-x^{1+A}}<0 \qquad \text{(A.39)}$$

We can rewrite the LHS of (A.39) as

$$\frac{(1+A)x^{1+A}(1-x^A)-x^A(x^{1+A}+A)(1-x^{1+A})}{(x^{1+A}+A)(1-x^{1+A})(1-x^A)}$$

Therefore, for (A.39), it is sufficient to show

$$x^A(x^{1+A}+A)(1-x^{1+A})>(1+A)x^{1+A}(1-x^A)$$

Notice that $1-x^{1+A}>1-x^A$, so it is sufficient to show $x^A(x^{1+A}+A)>(1+A)x^{1+A}$, or $x^{1+A}+A-(1+A)x>0$.

Consider $\frac{d}{dx}(x^{1+A}+A-(1+A)x)=(1+A)x^A-(1+A)=-(1+A)(1-x^A)<0$, so $x^{1+A}+A-(1+A)x$ decreases in x, and its minimum is reached at $x=1$ and the minimum is 0. Thus, $x^{1+A}+A-(1+A)x\geq 0$ and (A.39) is true.

Because of (A.39), we can rewrite (A.38)>0 as

$$\frac{\frac{\frac{1}{A^2}x^{1+A}}{1+\frac{1}{A}x^{1+A}}}{\frac{x^A}{1-x^A}-\frac{x^{1+A}}{1-x^{1+A}}-\frac{\frac{1}{A}x^{1+A}}{1+\frac{1}{A}x^{1+A}}}<-\ln x \qquad \text{(A.40)}$$

where the LHS can be rewritten as

$$\frac{(1-x^A)(1-x^{1+A})x}{A[(1+A)(1-x)-(1-x^{1+A})^2]}$$

Therefore, (A.40) is equivalent to

$$\frac{1-x^A}{A}\frac{1-x^{1+A}}{(1+A)(1-x)-(1-x^{1+A})^2}\leq-\frac{\ln x}{x}$$

which is true according to (A.33). ■

Claim A.3 $\frac{1+A}{A^2}\frac{1-x^A}{1-x^{1+A}}$ *decreases in* $A \in (0,2)$.

Proof. Differentiating the logarithm of the above expression, we obtain

$$\begin{aligned}
&\frac{d}{dA}\left(\ln(1+A) - 2\ln A + \ln(1-x^A) - \ln(1-x^{1+A})\right)\\
=\ & \frac{1}{1+A} - \frac{2}{A} + \frac{-x^A \ln x}{1-x^A} - \frac{-x^{1+A}\ln x}{1-x^{1+A}}\\
=\ & -\frac{2+A}{A(1+A)} - \ln x \frac{x^A(1-x)}{(1-x^A)(1-x^{1+A})}
\end{aligned} \tag{A.41}$$

There is a basic inequality of logarithmic function: $\ln(1+x) \geq \frac{x}{2}\frac{2+x}{x}$ for $x \in (-1,0)$, or equivalently, $-\ln x \leq \frac{1-x}{2}\frac{1+x}{x}$ for $x \in (0,1)$. Replacing x by x^A, we have $-\ln x \leq \frac{1-x^A}{2}\frac{1+x^A}{Ax^A}$. Substituting the inequality into (A.41) we have

$$\begin{aligned}
(A.41) \ &\leq\ -\frac{2+A}{A(1+A)} + \frac{1+x^A}{2A}\frac{1-x}{1-x^{1+A}}\\
&=\ -\frac{1}{A}\left(\frac{2+A}{1+A} - \frac{1+x^A}{2}\frac{1-x}{1-x^{1+A}}\right)
\end{aligned}$$

Notice $\frac{2+A}{1+A} > 1$, $\frac{1+x^A}{2} < 1$, and $\frac{1-x}{1-x^{1+A}} < 1$, so the above expression is negative, and so is (A.41). ■

Claim A.4

$$\frac{1+A}{A^2}\frac{1-(1-A)x^A}{1-x^{1+A}} > \frac{1+A'}{A'^2}\frac{1-(1-A')x^{A'}}{1-x^{1+A'}} \tag{A.42}$$

for $0 < A < A' < 2$.

Proof. Notice $\frac{1+A}{A} > \frac{1}{A} + 1 > \frac{1}{A'} + 1 = \frac{1+A'}{A'}$ and $\frac{x^A}{1-x^{1+A}} > \frac{x^{A'}}{1-x^{1+A'}}$, so

$$\frac{1+A}{A}\frac{x^A}{1-x^{1+A}} > \frac{1+A'}{A'}\frac{x^{A'}}{1-x^{1+A'}} \tag{A.43}$$

Consider

$$\begin{aligned}
\text{LHS of (A.42)} \ &=\ \frac{1+A}{A^2}\frac{1-x^A}{1-x^{1+A}} + \frac{1+A}{A}\frac{x^A}{1-x^{1+A}}\\
&>\ \frac{1+A'}{A'^2}\frac{1-x^{A'}}{1-x^{1+A'}} + \frac{1+A'}{A'}\frac{x^{A'}}{1-x^{1+A'}}\\
&=\ \text{RHS of (A.42)}
\end{aligned} \tag{A.44}$$

where the inequality is from Claim A.3 and (A.43). ∎

Claim A.5

$$\frac{\frac{1+A}{A}\frac{x^{1+A}}{1-x^{1+A}} - \frac{1+A'}{A'}\frac{x^{1+A'}}{1-x^{1+A'}}}{\frac{1+A}{A^2}\frac{1-(1-A)x^A}{1-x^{1+A}} - \frac{1+A'}{A'^2}\frac{1-(1-A')x^{A'}}{1-x^{1+A'}}} < x \tag{A.45}$$

Proof. Since the denominator is positive due to (A.42), we can rewrite (A.45) as

$$\frac{1+A}{A}\frac{x^{1+A}}{1-x^A} - \frac{1+A'}{A'}\frac{x^{A'}}{1-x^{1+A'}} < \frac{1+A}{A^2}\frac{1-(1-A)x^A}{1-x^{1+A}} - \frac{1+A'}{A'^2}\frac{1-(1-A')x^{A'}}{1-x^{1+A'}}$$

which is equivalent to

$$\frac{1+A}{A}\frac{x^{1+A}}{1-x^A} - \frac{1+A}{A^2}\frac{1-(1-A)x^A}{1-x^{1+A}} < \frac{1+A'}{A'}\frac{x^{A'}}{1-x^{1+A'}} - \frac{1+A'}{A'^2}\frac{1-(1-A')x^{A'}}{1-x^{1+A'}} \tag{A.46}$$

We have

$$\text{LHS of (A.46)} = 1\frac{(1+A)(1-x^A)}{A^2(1-x^{1+A})} \geq 1\frac{(1+A')(1-x^{A'})}{A'^2(1-x^{1+A'})} = \text{RHS of (A.46)} \tag{A.47}$$

where the inequality is from Claim A.3. ∎

Claim A.6

$$\frac{(1+A)^2}{A}\frac{x^A}{(1-x^{1+A})^2} > \frac{(1+A')^2}{A'}\frac{x^{A'}}{(1-x^{1+A'})^2} \tag{A.48}$$

for $0 < A < A' < 2$.

Proof. Notice the left hand side of (A.48) $= \frac{1+A}{A}\frac{(1+A)x^A}{1-x^{1+A}}\frac{1}{1-x^{1+A}}$, so it is sufficient to show $\frac{(1+A)x^A}{1-x^{1+A}}$ decreases in A. Its derivative w.r.t. A is

$$\begin{aligned}\frac{d}{dA}\left(\frac{(1+A)x^A}{1-x^{1+A}}\right) &= \frac{(x^A+(1+A)x^A\ln x)(1-x^{1+A})-(1+A)x^A(-x^{1+A})\ln x}{(1-x^{1+A})^2} \\ &= \frac{x^A}{(1-x^{1+A})^2}\left[(1+(1+A)\ln x)(1-x^{1+A})+(1+A)x^{1+A}\ln x\right]\end{aligned}$$

so it is sufficient to show

$$(1+(1+A)\ln x)(1-x^{1+A})+(1+A)x^{1+A}\ln x < 0 \tag{A.49}$$

which is equivalent to

$$1 - x^{1+A} < -\ln x^{1+A} \tag{A.50}$$

Recall $\ln x < x - 1$. Replacing x with x^{1+A}, we have $\ln x^{1+A} < x^{1+A} - 1$, so (A.50) holds and so does (A.48). ■

Claim A.7

$$x^A(1-x) + x^A(1-x^{1+A}) - \frac{1}{A}x^{1+A}(1-x^A) \geq 0 \tag{A.51}$$

Proof. The above inequality is equivalent to $(1-x)+(1-x^{1+A}) - \frac{1}{A}x(1-x^A) \geq 0$. The derivative of the left hand side w.r.t. x is

$$\begin{aligned}
& -1-(1+A)x^A - \frac{1}{A} + \frac{1}{A}(1+A)x^A \\
= & -\frac{1+A}{A}(1+x^A(A-1)) \\
\leq & -\frac{1+A}{A}(1+x^A(0-1)) \\
= & -\frac{1+A}{A}(1-x^A) < 0
\end{aligned}$$

where the first inequality is from $A \in (0,2)$. Therefore, $(1-x)+(1-x^{1+A}) - \frac{1}{A}x(1-x^A)$ decreases in x, so its minimum is $(1-1)+(1-1)-\frac{1}{A}(1-1) = 0$. ■

Lemma A.5 *Equation (A.32) has at most one solution in* $x \in (0,1)$.

Proof. It is sufficient to show the left hand side of (A.32) is increasing in x when (A.32) holds. Let

$$\begin{aligned}
\Delta_{A'} &= t - x^{1+A'}\left((1-A')\frac{t}{x} + A'\right) \\
\Delta_A &= t - x^{1+A}\left((1-A)\frac{t}{x} + A\right)
\end{aligned}$$

Then, the left hand side of (A.32) is $\frac{\Delta_{A'}}{\Delta_A}\frac{1-x^{1+A}}{1-x^{1+A'}}$. Consider

$$\frac{d}{dx}\ln\left(\frac{\Delta_{A'}}{\Delta_A}\frac{1-x^{1+A}}{1-x^{1+A'}}\right) = \frac{\Delta'_{A'}}{\Delta_{A'}} - \frac{\Delta'_A}{\Delta_A} - \frac{(1-x^{1+A'})'}{1-x^{1+A'}} + \frac{(1-x^{1+A})'}{1-x^{1+A}}$$

Therefore, to show the left hand side of (A.32) is increasing in x, it is sufficient to show

$$\frac{\Delta'_{A'}}{\Delta_{A'}} - \frac{(1-x^{1+A'})'}{1-x^{1+A'}} > \frac{\Delta'_A}{\Delta_A} - \frac{(1-x^{1+A})'}{1-x^{1+A}} \tag{A.52}$$

or equivalently

$$\begin{aligned}
& \Delta'_{A'}(1-x^{1+A'}) - \Delta_{A'}(1-x^{1+A'})' \\
> \; & \left(\Delta'_A(1-x^{1+A}) - \Delta_A(1-x^{1+A})'\right) \frac{\Delta_{A'}}{\Delta_A} \frac{1-x^{1+A}}{1-x^{1+A'}} \left(\frac{1-x^{1+A'}}{1-x^{1+A}}\right)^2 \\
= \; & \left(\Delta'_A(1-x^{1+A}) - \Delta_A(1-x^{1+A})'\right) \frac{\frac{1+A}{A^2}}{\frac{1+B}{B^2}} \left(\frac{1-x^{1+A'}}{1-x^{1+A}}\right)^2
\end{aligned}$$

where the equality is from (A.32). We can rewrite the above inequality as

$$\frac{1+A'}{A'^2} \frac{\Delta'_{A'}(1-x^{1+A'}) - \Delta_{A'}(1-x^{1+A'})'}{(1-x^{1+A'})^2} > \frac{1+A}{A^2} \frac{\Delta'_A(1-x^{1+A}) - \Delta_A(1-x^{1+A})'}{(1-x^{1+A})^2} \tag{A.53}$$

Substituting $\Delta'_{A'}$ and $\Delta_{A'}$ into (A.53), we have

$$\begin{aligned}
& \text{LHS of (A.53)} \\
= \; & \frac{1+A'}{A'^2} \frac{1}{(1-x^{1+A'})^2} \Big[\Big(-t(1-A')A'x^{A'-1} - A'(1+A')x^{A'} \Big) (1-x^{1+A'}) \\
& - \Big(t(1-(1-A')x^{A'}) - A'x^{1+A'} \Big) (1-x^{1+A'})' \Big] \\
= \; & -t \frac{1+A'}{A'^2} \frac{(1-A')A'x^{A'-1}(1-x^{1+A'}) + (1-(1-A')x^{A'})(1-x^{1+A'})'}{(1-x^{1+A'})^2} \\
& + \frac{1+A'}{A'^2} \frac{-A'(1+A')x^{A'}(1-x^{1+A'}) + A'x^{1+A'}(1-x^{1+A'})'}{(1-x^{1+A'})^2}
\end{aligned}$$

As a result, we can rewrite (A.53) as

$$\begin{aligned}
& t \Bigg[-\frac{1+A'}{A'^2} \frac{(1-A')A'x^{A'-1}(1-x^{1+A'}) + (1-(1-A')x^{A'})(1-x^{1+A'})'}{(1-x^{1+A'})^2} \\
& + \frac{1+A}{A^2} \frac{(1-A)Ax^{A-1}(1-x^{1+A}) + (1-(1-A)x^{A})(1-x^{1+A})'}{(1-x^{1+A})^2} \Bigg] \\
> \; & -\frac{1+A'}{A'^2} \frac{-A'(1+A')x^{A'}(1-x^{1+A'}) + A'x^{1+A'}(1-x^{1+A'})'}{(1-x^{1+A'})^2} \\
& + \frac{1+A}{A^2} \frac{-A(1+A)x^{A}(1-x^{1+A}) + Ax^{1+A}(1-x^{1+A})'}{(1-x^{1+A})^2}
\end{aligned}$$

which is equivalent to

$$t\left[-\frac{1+A'}{A'^2}\frac{(1-A')A'x^{A'-1}-(1+A')x^{A'}+(1-A')x^{2A'}}{(1-x^{1+A'})^2}\right.$$
$$\left.+\frac{1+A}{A^2}\frac{(1-A)Ax^{A-1}-(1+A)x^{A}+(1-A)x^{2A}}{(1-x^{1+A})^2}\right]$$
$$> \frac{(1+A')^2}{A'}\frac{x^{A'}}{(1-x^{1+A'})^2}-\frac{(1+A)^2}{A}\frac{x^{A}}{(1-x^{1+A})^2} \tag{A.54}$$

Let

$$\Phi = -\frac{1+A'}{A'^2}\frac{(1-A')A'x^{A'-1}-(1+A')x^{A'}+(1-A')x^{2A'}}{(1-x^{1+A'})^2}$$
$$+\frac{1+A}{A^2}\frac{(1-A)Ax^{A-1}-(1+A)x^{A}+(1-A)x^{2A}}{(1-x^{1+A})^2}$$

so (A.54) becomes

$$t\Phi > \frac{(1+A')^2}{A'}\frac{x^{A'}}{(1-x^{1+A'})^2}-\frac{(1+A)^2}{A}\frac{x^{A}}{(1-x^{1+A})^2}$$

Recall that we want to show (A.54) is true when (A.32) holds. From (A.32), we can solve for t and obtain

$$t = \frac{\frac{1+A}{A}\frac{x^{1+A}}{1-x^{1+A}}-\frac{1+A'}{A'}\frac{x^{1+A'}}{1-x^{1+A'}}}{\frac{1+A}{A^2}\frac{1-(1-A)x^{A}}{1-x^{1+A}}-\frac{1+A'}{A'^2}\frac{1-(1-A')x^{A'}}{1-x^{1+A'}}} \tag{A.55}$$

Thus, it remains to show (A.54) is true when (A.55) holds.

Recall that (A.48) implies the RHS of (A.54) to be negative. Then, if $\Phi > 0$, LHS of (A.54) $> 0 >$ RHS of (A.54), so (A.32) is true.

If $\Phi < 0$, LHS of (A.54) $= t\Phi > x\Phi$ because of (A.45), so it remains to show $x\Phi >$ RHS of (A.54). That is,

$$-\frac{1+A'}{A'^2}\left((1-A')A'x^{A'}-(1+A')x^{A'+1}+(1-A')x^{2A'+1}\right)(1-x^{1+A})^2$$
$$+\frac{1+A}{A^2}\left((1-A)Ax^{A}-(1+A)x^{A+1}+(1-A)x^{2A+1}\right)(1-x^{1+A'})^2$$
$$> \frac{(1+A')^2}{A'}x^{A'}(1-x^{1+A})^2-\frac{(1+A)^2}{A}x^{A}(1-x^{1+A'})^2$$

Collecting terms, we can rewrite the above inequality as

$$
\begin{aligned}
&(1-x^{1+A'})^2 x^A \left[\frac{2+2A}{A} - \left(\frac{1+A}{A}\right)^2 x + \frac{1-A^2}{A^2} x^{1+A}\right] \\
> \;& (1-x^{1+A})^2 x^{A'} \left[\frac{2+2A'}{A'} - \left(\frac{1+A'}{A'}\right)^2 x + \frac{1-A'^2}{A'^2} x^{1+A'}\right]
\end{aligned}
$$

If we separate A and A', the above inequality becomes

$$
\begin{aligned}
&\frac{1+A}{A} \frac{x^A(1-x) + x^A(1-x^{1+A}) - \frac{1}{A} x^{1+A}(1-x^A)}{(1-x^{1+A})^2} \\
> \;& \frac{1+A'}{A'} \frac{x^{A'}(1-x) + x^{A'}(1-x^{1+A'}) - \frac{1}{A'} x^{1+A'}(1-x^{A'})}{(1-x^{1+A'})^2}
\end{aligned} \tag{A.56}
$$

Notice that each side of (A.56) is a product of two terms, and the second term is positive due to (A.51). Moreover, $\frac{1+A}{A} > \frac{1+A'}{A'}$ and $\frac{1}{1-x^{1+A}} > \frac{1}{1-x^{1+A'}}$, so it is sufficient to show

$$
\frac{x^A(1-x) + x^A(1-x^{1+A}) - \frac{1}{A} x^{1+A}(1-x^A)}{(1-x^{1+A})} > \frac{x^{A'}(1-x) + x^{A'}(1-x^{1+A'}) - \frac{1}{A'} x^{1+A'}(1-x^{A'})}{(1-x^{1+A'})}
$$

which is equivalent to

$$
\frac{x^A(1-x)}{(1-x^{1+A})} + x^A - \frac{1}{A} x^{1+A} \frac{1-x^A}{1-x^{1+A}} > \frac{x^{A'}(1-x)}{(1-x^{1+A'})} + x^{A'} - \frac{1}{A'} x^{1+A'} \frac{1-x^{A'}}{1-x^{1+A'}}
$$

Since $x^A > x^{A'}$, it is sufficient to show

$$
\frac{x^A(1-x)}{(1-x^{1+A})} - \frac{1}{A} x^{1+A} \frac{1-x^A}{1-x^{1+A}} > \frac{x^{A'}(1-x)}{(1-x^{1+A'})} - \frac{1}{A'} x^{1+A'} \frac{1-x^{A'}}{1-x^{1+A'}} \tag{A.57}
$$

Notice

$$
\begin{aligned}
\text{LHS of (A.57)} &= x^A \frac{1-x}{1-x^{1+A}} - \frac{1}{A} x^{1+A} \left(1 - x^A \frac{1-x}{(1-x^{1+A})}\right) \\
&= \left(\frac{1}{A} x^{1+A} + 1\right) \frac{-(1-x^A)}{1-x^{1+A}} + 1
\end{aligned}
$$

so (A.57) is equivalent to

$$
\left(\frac{1}{A} x^{1+A} + 1\right) \frac{1-x^A}{1-x^{1+A}} < \left(\frac{1}{A'} x^{1+A'} + 1\right) \frac{1-x^{A'}}{1-x^{1+A'}}
$$

which is true according to Claim A.2. ■

C.3 Assumptions 1 and 2 for Other Distribution Families

For F within the following distribution families, $L_F(v)$ and $K_F(v)$ do not have closed-form expressions, so we verify Assumptions 1 and 2 numerically. Specifically, we have

$$\begin{aligned}
\frac{dq_t}{dv} &= -\frac{t}{v^2}\frac{1}{F^{n-1}(q_t)+(n-1)q_tF^{n-2}(q_t)f(q_t)} \\
L_F'(v) &= F^n(q_t)-(1-v)\frac{dF^n(q_t)}{dv}-1 \\
&= -(1-F^n(q_t))-(1-v)nF^{n-1}(q_t)f(q_t)\frac{dq_t}{dv} \\
K_F'(v) &= \int_{q_t}^{w_F} s(1-F^n(s))dF^{n-1}(s)-vq_t(1-F^n(q_t))\frac{dF^{n-1}(q_t)}{dv} \\
&= \int_{q_t}^{w_F} s(1-F^n(s))dF^{n-1}(s)-vq_t(1-F^n(q_t))(n-1)F^{n-1}(q_t)f(q_t)\frac{dq_t}{dv}
\end{aligned}$$

Substituting $\frac{dq_t}{dv}$ into $L_F'(v)$ and $K_F'(v)$, we can rewrite them as a function of q_r, v and n. Then, for Assumption 1, we can compute the value of $\frac{L_F'(v)}{K_F'(v)}$ for each v and verify that it is strictly decreasing in v whenever $\frac{L_F'(v)}{K_F'(v)} > \lim_{v\to\infty}\frac{L_F'(v)}{K_F'(v)}$, where

$$\lim_{v\to\infty}\frac{L_F'(v)}{K_F'(v)} = \frac{1}{\int_0^{w_F} s(1-F^n(s))dF^{n-1}(s)} \tag{A.58}$$

according to (A.4) and (A.7).

- **Pareto Distributions** Consider the c.d.f. $F(q;\alpha)=1-(1+q)^\alpha$ for $q\in[0,+\infty)$ with parameter $\alpha<-1$. We focus on $\alpha<-1$ to ensure that the mean $-\frac{1}{1+\alpha}$ is finite, and show that Assumption 1 holds for a wide range of parameters given by $(n,r,\alpha)\in\{2,...,50\}\times(0,2)\times(-11,-1.1)$. The above parameter set is selected such that n varies from 2 to 50, where with $n=50$, the distribution of the highest idea quality $F^{50}(q)$ is heavily skewed towards high q values; r varies from 0 to 2, which is twice the gross gain when $\lambda=0$; and α varies from -11 to -1.1, which gives a range of the mean from 0.1 to 10. Our programs are readily extendable to even larger sets of parameters.

- **Exponential Distributions** Consider the c.d.f. $F(q;\alpha)=1-e^{-\alpha q}$ for $q\in[0,+\infty)$ and parameter $\alpha>0$. We show that Assumption 1 holds for a wide range of parameters given by $(n,r,\alpha)\in\{2,...,50\}\times(0.1,2)\times(0.1,10)$.

- **Log-normal Distributions** Let $F(q;\mu,\sigma^2)$ for $q\geq 0$ be the c.d.f. of log-normal distribution with mean $\mu>0$ and variance $\sigma^2>0$. We show that Assumption 1 holds for a wide range of parameters given by $(n, r, \mu, \sigma) \in \{2, ..., 50\} \times (0.1, 2) \times (0.1, 10) \times (0.1, 10)$.

Recall that for Assumption 2, we need to verify that $\frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_G(v)}{K'_G(v)}$ cross at most once over the range of v such that $\frac{L'_F(v)}{K'_F(v)} > \lim_{v\to\infty} \frac{L'_F(v)}{K'_F(v)}$ and $\frac{L'_G(v)}{K'_G(v)} > \lim_{v\to\infty} \frac{L'_G(v)}{K'_G(v)}$.

- **Pareto Distributions** Consider $F(q) = 1-(1+q)^{\alpha}$ and $G(q) = 1-(1+q)^{\alpha'}$ with $\alpha' > \alpha$. We show that Assumption 2 holds for a wide range of parameters given by $(n, r) \in \{2, ..., 50\} \times (0.1, 2)$ and $-11 < \alpha < \alpha' < -1.1$.

- **Exponential Distributions** Consider the c.d.f. $F(q) = 1 - e^{-\alpha q}$ and $G(q) = 1 - e^{-\alpha' q}$ for $q \in [0, +\infty)$ and parameter $\alpha > 0$. We show that Assumption 2 holds for a wide range of parameters given by $(n, r, \alpha) \in \{2, ..., 50\} \times (0.1, 2) \times (0.1, 10)$.

- **Log-normal Distributions** Let $F(q; \mu, \sigma^2)$ for $q \geq 0$ be the c.d.f. of log-normal distribution with mean $\mu > 0$ and variance $\sigma^2 > 0$. We show that Assumption 2 holds for a wide range of parameters given by $(n, r, \mu, \sigma) \in \{2, ..., 50\} \times (0.1, 2) \times (0.1, 10) \times (0.1, 10)$.

D Equivalence of $F \prec G$ and $F \prec_{FOSD} G$

Recall that Lemma 4 shows that $F \prec G$ implies $F \prec_{FOSD} G$. We show below that $F \prec_{FOSD} G$ implies $F \prec G$ for many distribution families. First, note that for F and G within the following distribution families, $\hat{F}(q)$ and $\hat{G}(q)$ have closed-form expressions.

- **Uniform Distributions** Consider $F(q) = q/w_F$ and $G(q) = q/w_G$ with $0 < w_F < w_G$. The supports are $[0, w_F]$ for F and $[0, w_G]$ for G. Then, $\phi_F(q) = q(q/w_F)^{n-1}$, so $\phi_F^{-1}(x) = (w_F)^{\frac{n-1}{n}} x^{\frac{1}{n}}$ and $\hat{F}(x) = (x/w_F)^{1/n}$. Similarly, $\hat{G}(x) = (x/w_G)^{1/n}$. Then, it is straightforward to verify $F \prec G$.

- **Power Function Distributions** Consider $F(q) = q^{\alpha}$ and $G(q) = q^{\alpha'}$ with $q \in [0, 1]$ and $0 < \alpha < \alpha'$. Then, $\phi_F(q) = q^{(n-1)\alpha+1}$, so $\phi_F^{-1}(x) = x^{\frac{1}{(n-1)\alpha+1}}$ and $\hat{F}(x) = x^{\frac{\alpha}{(n-1)\alpha+1}}$. Similarly, $\hat{G}(x) = x^{\frac{\alpha'}{(n-1)\alpha'+1}}$. Thus, $F \prec G$.

For F and G within the following distribution families, $\hat{F}(q)$ and $\hat{G}(q)$ do not have closed-form expressions, so we verify $F \prec G$ numerically.

- **Pareto Distributions** Consider the c.d.f. $F(q; \alpha) = 1 - (1 + q)^{\alpha}$ for $q \in [0, +\infty)$ with parameter $\alpha < -1$. We focus on $\alpha < -1$ to ensure that the mean $-\frac{1}{1+\alpha}$ is finite. Then, $F(\cdot;\alpha') \prec_{FOSD} F(\cdot;\alpha'')$ if and only if $\alpha' \leq \alpha''$. Notice that the density of effective ideas also depends on α, so we write it as $\hat{f}(x;\alpha)$. A sufficient condition for $F(\cdot;\alpha') \prec F(\cdot;\alpha'')$ is $\frac{\partial^2 log\hat{f}(x;\alpha)}{\partial x \partial \alpha} > 0$ for all $\alpha \in (\alpha', \alpha'')$ and for all $x > 0$. We verify this condition for $-11 \leq \alpha \leq -1.1$ and $2 \leq n \leq 50$. Thus, for those α and n values, the first order stochastic dominance and the stochastic order in Definition 2 are equivalent.

- **Exponential Distributions** Consider the c.d.f. $F(q;\alpha) = 1 - e^{-\alpha q}$ for $q \in [0, +\infty)$ and parameter $\alpha > 0$. As above, $F(\cdot;\alpha') \prec_{FOSD} F(\cdot;\alpha'')$ if and only if $\alpha' \leq \alpha''$. We can verify the sufficient condition that $\frac{\partial^2 log\hat{f}(x;\alpha)}{\partial x \partial \alpha} > 0$ for $0.1 \leq \alpha \leq 10$ and $2 \leq n \leq 50$. Thus, for those α and n values, the first order stochastic dominance and the stochastic order in Definition 2 are equivalent.

- **Log-normal Distributions** Let $F(q;\mu,\sigma^2)$ for $q \geq 0$ be the c.d.f. of log-normal distribution with mean $\mu > 0$ and variance $\sigma^2 > 0$. For any fixed σ^2, we have $F(\cdot;\mu',\sigma^2) \prec_{FOSD} F(\cdot;\mu'',\sigma^2)$ if and only if $\mu' \leq \mu''$. As above, we can verify the sufficient condition that $\frac{\partial^2 log\hat{f}(x;\mu,\sigma^2)}{\partial x \partial \mu} > 0$ for $0.1 \leq \mu \leq 10$, $0.1 \leq \sigma \leq 10$ and $2 \leq n \leq 50$. Thus, for those μ and n values, the first order stochastic dominance and the stochastic order in Definition 2 are equivalent.

E Proofs of the Properties stated in Footnote 18

We prove the properties through Lemmas A.6-A.9.

Lemma A.6 *If $F \prec G$, then $q_t < q'_t$ and $F(q_t) > G(q'_t)$.*

Proof. The definition of $\hat{F} \prec_{LR} \hat{G}$ implies $\hat{F}(q) \geq \hat{G}(q)$, which combined with Lemma 2 implies $F(q) \geq G(q)$. Therefore, we have $\phi_F \geq \phi_G$, so $q_t \leq q'_t$. Since (2) and (8) imply $q_t F^n(q_t) = q'_t G^n(q'_t)$, inequality $q_t \leq q'_t$ implies $F(q_t) \geq G(q'_t)$. ■

Lemma A.7 $L'_F(1) > L'_G(1)$.

Proof. Suppose $v = 1$. Equation (A.6) implies $L'_G(v) - L'_F(v) = G^n(q'_t) - F^n(q_t) < 0$, where the inequality comes from Lemma A.6. ■

Lemma A.8 *$F \prec G$ implies that $G(q'_t)/F(q_t)$ decreases in v.*

Proof.

$$\begin{aligned}
\frac{d}{dv}\left(\log\frac{G(q'_t)}{F(q_t)}\right) &= \frac{d\log G(q'_t)}{dv} - \frac{d\log F(q_t)}{dv} \\
&= \frac{d\log \hat{G}(t/v)}{d(t/v)}\left(-\frac{t}{v^2}\right) - \frac{d\log \hat{F}(t/v)}{d(t/v)}\left(-\frac{t}{v^2}\right) \\
&= \frac{\hat{g}(t/v)}{\hat{G}(t/v)}\left(-\frac{t}{v^2}\right) - \frac{\hat{f}(t/v)}{\hat{F}(t/v)}\left(-\frac{t}{v^2}\right) \\
&< 0
\end{aligned}$$

where the inequality comes from the reverse hazard rate dominance in the definition of $F \prec G$. ■

Lemma A.9 *Suppose $F \prec G$ and F and G have a common support $[0, 1]$. Then, $L'_F(t) < L'_G(t)$.*

Proof. If $v = t$, equation (2) becomes $q_t F^{n-1}(q_t) = 1$. Since the LHS is strictly increasing in q_t, this equation has a unique solution $q_t = 1$. Then, the common support $[0, 1]$ implies

$$q_t = q'_t = F(q_t) = G(q'_t) = 1 \tag{A.59}$$

If $v = t$, we also have

$$\frac{dF^n(q_t)}{dv} = \frac{d\hat{F}^n(t/v)}{d(t/v)}\left(-\frac{t}{v^2}\right) = -\frac{n}{t}\hat{f}(1) > -\frac{n}{t}\hat{g}(1) = \frac{dG^n(q'_t)}{dv} \tag{A.60}$$

where the second equality comes from $v = t$ and the inequality comes from $\hat{f}(1) < \hat{g}(1)$ implied by the reverse hazard rate dominance in the definition of $F \prec G$.

Therefore,

$$\begin{aligned}
L'_G(v) - L'_F(v) &= [G^n(q'_t) - F^n(q_t)] - (1 - v)\left(\frac{dG^n(q'_t)}{dv} - \frac{dF^n(q_t)}{dv}\right) \\
&= (1 - 1) - (1 - v)\left(\frac{dG^n(q'_t)}{dv} - \frac{dF^n(q_t)}{dv}\right) \\
&> 0
\end{aligned}$$

where the second equality comes from (A.59) and the inequality comes from (A.60). ■

F Proofs in Section 6.1

We denote the optimal prizes for nonlinear benefits as $V_{F,B}$ and $V_{G,B}$ for distributions F and G, respectively. The following result generalizes Lemma A.3 to the case of nonlinear benefits.

Lemma A.10 *Under Assumption* $1'$*, if* $F \prec G$*, then* $0 < \bar{\lambda}_{G,B} < \bar{\lambda}_{F,B} < +\infty$*, where* $\bar{\lambda}_{F,B} = \inf\{\lambda | V_{F,B}(\lambda) = v_{max}\}$ *and* $\bar{\lambda}_{G,B} = \inf\{\lambda | V_{G,B}(\lambda) = v_{max}\}$.

Proof. First we show $\lim_{v\to\infty} K'_{F,B}(v) = \underline{b} \lim_{v\to\infty} K'_F(v)$. Notice that the definition of $K_{F,B}$ implies

$$K'_{F,B}(v) = \int_{q_t}^{w_F} B'\left(v \int_{q_t}^{q} s dF^{n-1}(s)\right) \left(\int_{q_t}^{q} s dF^{n-1}(s) - v q_t \frac{dF^{n-1}(q_t)}{dv}\right) dF^n(q)$$

so

$$\begin{aligned}
\lim_{v\to\infty} K'_{F,B}(v) &= \underline{b} \int_0^{w_F} \left(\int_0^q s dF^{n-1}(s) + \lim_{v\to+\infty} v q_t \frac{dF^n(q_t)}{dv}\right) dF^n(q) \\
&= \underline{b} \int_0^{w_F} \left(\int_0^q s dF^{n-1}(s)\right) dF^n(q) \\
&= \underline{b} \int_0^{w_F} s(1 - F^n(s)) dF^{n-1}(s) \qquad \text{(A.61)}
\end{aligned}$$

where the second equation comes from $\lim_{v\to\infty} v q_t dF^n(q_t)/dv = 0$, and the last from changing the order of integration. In addition, (A.4) implies that

$$\begin{aligned}
\lim_{v\to\infty} K'_F(v) &= \int_0^{w_F} s(1 - F^n(s)) dF^{n-1}(s) - \lim_{v\to\infty} v q_t (1 - F^n(q_t)) \frac{dF^{n-1}(q_t)}{dv} \\
&= \int_0^{w_F} s(1 - F^n(s)) dF^{n-1}(s) \qquad \text{(A.62)}
\end{aligned}$$

where the second equation is also from $\lim_{v\to\infty} v q_t dF^n(q_t)/dv = 0$. Hence, (A.61) and (A.62) imply that $\lim_{v\to\infty} K'_{F,B}(v) = \underline{b} \lim_{v\to\infty} K'_F(v)$.

Next, we prove the lemma. Notice that the proof of Lemma A.3 only uses $\lim_{v\to\infty} K'_F(v)$. Because $\lim_{v\to\infty} K'_{F,B}(v) = \underline{b} \lim_{v\to\infty} K'_F(v)$, the proof of Lemma A.3 applies to nonlinear benefit function B as well. ■

Proposition A.1 *Under Assumption* $1'$*, there is a unique optimal prize* $V_{F,B}(\lambda)$ *for any* $\lambda \neq \bar{\lambda}_{F,B}$*. Moreover,* $V_{F,B}(\lambda)$ *is weakly increasing in* λ.

The proof is the same as that of Proposition 1 and is therefore omitted. The following result generalizes Proposition 2 to nonlinear benefit functions using a similar kind of argument.

Proposition A.2 *Under Assumptions* $1'$ *and* $2'$*, if* $F \prec G$*, there exists* $\hat{\lambda} > 0$ *such that*

i) $V_{G,B}(\lambda) < V_{F,B}(\lambda)$ *if* $\lambda < \hat{\lambda}$

ii) $V_{G,B}(\lambda) \geq V_{F,B}(\lambda)$ *if* $\lambda > \hat{\lambda}$

Proof. If $\lambda = 0$, the expected profit is $\Pi_{F,B}(v) = L_F(v)$, which is the same as the expected profit in Section 4. Therefore, following the same argument as in the proof of Proposition 2, we have $V_F(\lambda) > V_G(\lambda)$ if λ is small.

Replacing Lemma A.3 with Lemma A.10, we can follow the same arguments as in Steps II and III of the proof of Proposition 2 to prove this proposition. ■

Assumptions $1'$ and $2'$ ensure that the optimal prize is unique and that there is a unique $\hat{\lambda}$ at which the order of the optimal prizes switches. Without Assumption $1'$, there may be multiple optimal prizes. Then, let $V_{F,B}(\lambda)$ and $V_{G,B}(\lambda)$ represent the set of optimal prizes. The following proposition states that a result similar to Proposition 3 holds. The proof is the same as that of Proposition 3, so we omit it here.

Proposition A.3 *If* $F \prec G$*, there exist* $\hat{\lambda}' \geq \hat{\lambda} > 0$ *such that*

i) $V_{G,B}(\lambda) \leq V_{F,B}(\lambda)$ *if* $\lambda < \hat{\lambda}$

ii) $V_{G,B}(\lambda) \geq V_{F,B}(\lambda)$ *if* $\lambda > \hat{\lambda}'$

G Proofs in Section 6.2

Lemma A.11 *A reservation performance* $r < t$ *is never optimal.*

Proof. Suppose the seeker chooses $r < t$. The seeker's expected payoff is

$$\int_{q_t}^{w_F} [1 + \lambda(\beta(q) - t)] dF^n(q) - v(1 - F^n(q_r))$$

The first term is her gross profit from the best performance and the second term is the cost of giving up the prize v. Note that the solvers with ideas $q_i \in [r, t)$ choose performances above r, but their performances are worthless to the seeker because they are still below t. Thus, the integration in the first term is for $q \geq q_t$.

Note that q_r decreases as r increases. Thus, the gross profit (the first term) remains the same, but the cost (the second term) is lower. This means an increase in r leads to an increase in the seeker's profit. Hence, $r < t$ is never optimal. ■

Derivation of equation (10) According to the equilibrium strategies, the expected highest performance is

$$\begin{aligned}
\int_{q_r}^{w_F} \beta(q) dF^n(q) &= \int_{q_r}^{w_F} \left(r + v \int_{q_r}^{q} s dF^{n-1}(s) \right) dF^n(q) \\
&= \int_{q_r}^{w_F} \left(v q_r F^{n-1}(q_r) + v \int_{q_r}^{q} s dF^{n-1}(s) \right) dF^n(q) \\
&= \int_{q_r}^{w_F} \left(v q F^{n-1}(q) - v \int_{q_r}^{q} F^{n-1}(s) ds \right) dF^n(q) \\
&= v \int_{q_r}^{w_F} \left[q F^{n-1}(q)(F^n)'(q) - F^{n-1}(q)(1 - F^n(q) \right] dq \\
&= v \int_{q_r}^{w_F} F^{n-1}(q) \left(q - \frac{1 - F^n(q)}{(F^n)'(q)} \right) dF^n(q)
\end{aligned}$$

where the second equation is from (9), the third from integration by parts, and the fourth from changing the order of integration. Substituting the above expression into the seeker's expected profit, we can rewrite it as in (10).

Lemma A.12 *If $\lambda = 0$, the optimal reservation performance level is t.*

Proof. If $\lambda = 0$, the expected profit is $L_F(v, q_r) = (1 - v)(1 - F^n(q_r))$ with $q_r \geq q_t$, which is decreasing in q_r. Therefore, if (v, r) is optimal, it must ensure $q_r = q_t$. This means that $r = t$ whatever the optimal v is. ■

Proof of Proposition 4. We prove in three steps.

Step I. We prove part i). Specifically, Lemma A.12 implies that $\partial \Pi_F(v, q_r)/\partial q_r < 0$ if $\lambda = 0$. Because $\partial \Pi_F(v, q_r)/\partial q_r$ is continuous in λ, there exists $\lambda' > 0$ such that $\partial \Pi_F(v, q_r)/\partial q_r < 0$ for $\lambda \in [0, \lambda']$. Then, for these λ values, the optimal q_r is at the lower bound q_t, which implies the optimal r equals t. Hence, Proposition 1 implies that $V_F(\lambda)$ is monotone non-decreasing in λ if $\lambda \leq \lambda'$.

Step II. The optimal q_r is bounded away from w_F as $\lambda \to +\infty$. Suppose not. That is, suppose that $q_r \to w_F$ as $\lambda \to +\infty$. Then,

$$\begin{aligned}
& \lim_{\lambda \to +\infty} \frac{\int_{q_r}^{w_F} \left[\lambda F^{n-1}(q) \left(q - \frac{1 - F^n(q)}{(F^n)'(q)} \right) \right] dF^n(q)}{1 - F^n(q_r)} \\
\geq & \lim_{\lambda \to +\infty} \frac{\lambda F^{n-1}(q_r) \left(q_r - \frac{1 - F^n(q_r)}{(F^n)'(q_r)} \right) \int_{q_r}^{w_F} dF^n(q)}{1 - F^n(q_r)} \\
= & \lim_{\lambda \to +\infty} \lambda F^{n-1}(q_r) \left(q_r - \frac{1 - F^n(q_r)}{(F^n)'(q_r)} \right) \\
= & +\infty
\end{aligned}$$

where the inequality follows from Assumption 3 and the last equality follows from the assumption that $q_r \to w_F$ as $\lambda \to +\infty$ and $\lim_{q \to w_F} \left(q - \frac{1-F^n(q)}{(F^n)'(q)}\right) > 0$ from Assumption 3.

Recall that the seeker chooses (v, q_r) and their marginal profits are

$$\frac{\partial \Pi_F(v, q_r)}{\partial v} = \int_{q_r}^{w_F} \left[\lambda F^{n-1}(q)\left(q - \frac{1-F^n(q)}{(F^n)'(q)}\right) - 1\right] dF^n(q) \tag{A.63}$$

$$\frac{\partial \Pi_F(v, q_r)}{\partial q_r} = -\left\{1 + \lambda\left[vF^{n-1}(q_r)\left(q_r - \frac{1-F^n(q_r)}{(F^n)'(q_r)}\right) - t\right] - v\right\}(F^n)'(q_r) \tag{A.64}$$

Then, if λ is large enough, (A.63) implies that $\partial\Pi_F(v, q_r)/\partial v > 0$ for any v, so $v = v_{max}$. Recall that Assumption 3 implies that $\lim_{q \to w_F}\left(q - \frac{1-F^n(q)}{(F^n)'(q)}\right) > 0$ and denote the limit as $z > 0$. Then, if λ is large enough, (A.64) implies that $\partial\Pi_F(v, q_r)/\partial q_r$ for large enough q_r has the same sign as

$$-\lambda\left[v_{max} \lim_{q_r \to w_F} F^{n-1}(q_r)\left(q_r - \frac{1-F^n(q_r)}{(F^n)'(q_r)}\right) - t\right] = -\lambda\left[v_{max}z - t\right] < 0$$

where the equality follows from the large v_{max}. Thus, q_r is bounded away from w_F as $\lambda \to +\infty$.

Step III. If λ is large enough, the optimal is v_{max} and the optimal r is weakly decreasing in λ. To see this, notice that q_r is bounded away from w_F, so for large enough λ, $\partial\Pi_F(v, q_r)/\partial v > 0$ for all v. Hence, the optimal prize is v_{max}. With $v = v_{max}$ in (A.64), a higher λ shifts $\partial\Pi_F(v, q_r)/\partial q_r$ downwards, which results in a lower q_r. Intuitively, because v cannot be increased further, we need to lower r in order to increase the profits. ■

Lemma A.13 *If λ is sufficiently large, the optimal reservation performance level r is higher than threshold t.*

Proof. We know from above that if λ is large enough, the optimal prize is v_{max}. With $v = v_{max}$, the marginal profit in (A.64) at $q_r = q_t$ is

$$\frac{\partial \Pi_F(v_{max}, q_t)}{\partial q_r} = \left\{v_{max} - 1 + \lambda v_{max}F^{n-1}(q_t)\frac{1-F^n(q_t)}{(F^n)'(q_t)}\right\}(F^n)'(q_t) > 0$$

which means the optimal $r > t$ whenever optimal $v = v_{max}$. ■

Proof of Proposition 5. If λ is small enough, the optimal $r = t$, so Proposition 2 implies part i).

Notice that Lemma A.13 implies that if λ is large enough, the optimal $r > t$. Therefore, the optimal q_r satisfies the first order condition $\partial \Pi_F(v_{max}, q_r)/\partial q_r = 0$, which is equivalent to

$$F^{n-1}(q_r)\left(q_r - \frac{1 - F^n(q_r)}{(F^n)'(q_r)}\right) = \frac{1}{\lambda} - \frac{1}{\lambda v_{max}} + \frac{t}{v_{max}} \tag{A.65}$$

The left hand side of the above equation is increasing in q_r due to Assumption 3, so the above equation has a unique solution. Let $q_F(\lambda)$ be the solution. Then, the corresponding reservation performance is $R_F(\lambda) = v_{max} q_F(\lambda) F^{n-1}(q_F(\lambda))$. As $\lambda \to +\infty$, (A.65) becomes (11), and $q_F(\lambda)$ decreases and converges to q_F which is the solution of (11). As a result, $\lim_{\lambda \to +\infty} R_F(\lambda) = v_{max} q_F F^{n-1}(q_F)$. Thus, part ii) of the proposition is also true. ■

H Proofs in Section 6.3

Lemma A.14 *There is at most one q_t such that*

$$v^1 F^{n-1}(q_t) + v^2(n-1)(1 - F(q_t))F^{n-2}(q_t) = t/q_t \tag{A.66}$$

Proof. First, we consider the case with $v^1 = v^2$. Then, we can rewrite the left hand side of (A.66) as $v^2 F^{n-2}(q_t)((n-1) - (n-2)F(q_t))$. Differentiating the logarithm of the LHS w.r.t. q gives

$$(n-2)\frac{f(q_t)}{F(q_t)}\left(1 - \frac{1}{(n-1) - (n-2)F(q_t)}\right) \tag{A.67}$$

Notice that $1 - \frac{1}{(n-1)-(n-2)F(q_t)} > 0$ is equivalent to $F(q_t) < 1$, so (A.67) is positive. Thus, $v^2 F^{n-2}(q_t)((n-1) - (n-2)F(q_t))$ is increasing in q_t.

Next, we consider the general case with $v^1 \geq v^2 \geq 0$. We can rewrite the LHS of (A.66) as

$$(v^1 - v^2)F^{n-1}(q_t) + v^2 F^{n-2}(q_t)((n-1) - (n-2)F(q_t)) \tag{A.68}$$

where the first term is increasing in q_t and so is the second term due to the first step. Therefore, the LHS of (A.66) increases in q_t, and (A.66) has at most one solution. ■

Lemma A.15 *Given two prizes $v^1 \geq v^2$, the solver's equilibrium strategy is*

$$\beta_t(q) = \begin{cases} t + v^1 A_t(q) + v^2 B_t(q) & \text{if } q \geq q_t \\ 0 & \text{otherwise} \end{cases} \tag{A.69}$$

where q_t is the unique solution to (A.66) and

$$B_t(q) = (n-1)\left(\int_{q_t}^{q} s dF^{n-2}(s) - \int_{q_t}^{q} s dF^{n-1}(s)\right)$$

Proof. Similar to the case with a single prize, if a solver's idea quality is too low, he chooses to not participate. Moreover, the highest idea quality for non-participants, q_t, makes a solver indifferent between non-performance and the threshold t. That is

$$v^1 F^{n-1}(q_t) + v^2(n-1)(1-F(q_t))F^{n-2}(q_t) - \frac{t}{q_t} = 0$$

which can be rewritten as (A.66).

As a result, $\beta_t(q) = 0$ for $q < q_t$. It remains to characterize $\beta_t(q)$ for $q \geq q_t$. Assume β_t is differentiable for $q > q_t$. A solver's problem with two prizes $v^1 \geq v^2 \geq 0$ becomes

$$\max_x v^1 F^{n-1}(\beta_t^{-1}(x)) + v^2(n-1)(1-F(\beta_t^{-1}(x)))F^{n-2}(\beta_t^{-1}(x)) - \frac{x}{q}$$

Letting $y = \beta^{-1}(x)$ and substituting it into the first order condition, we obtain an analogue of (A.1):

$$v^1 y \frac{dF^{n-1}(y)}{dx} + v^2(n-1)y\left(-\frac{dF(y)}{dx}F^{n-2}(y) + (1-F(y))\frac{dF^{n-2}(y)}{dx}\right) = 1$$

which is also a differential equation with separated variables. Therefore, following the same way to solve (A.1), we have (A.69).

We can also verify that $\beta_t'(q) > 0$ for $q > q_t$. To see this, notice that

$$\begin{aligned}
\beta_t'(q) &= v^1 q (F^{n-1})'(q) + v^2 q(n-1)\left[(F^{n-2})'(q) - (F^{n-1})'(q)\right] \\
&= (v^1 - v^2) q (F^{n-1})'(q) + v^2 q\left[(n-1)(F^{n-2})'(q) - (n-2)(F^{n-1})'(q)\right] \\
&= (v^1 - v^2) q (F^{n-1})'(q) + v^2 q(n-1)(n-2)F^{n-3}(q) f(q)(1-F(q)) \\
&> 0
\end{aligned}$$

■

Before moving on to the next lemma, we introduce some new notation. Let $\alpha \in [0, 1/2]$ be the allocation to the second prize, so $v^1 = \bar{v}(1-\alpha)$ and $v^2 = \bar{v}\alpha$. The seeker's expected profit is $\Pi_F(\bar{v}, \alpha) = L_F(\bar{v}, \alpha) + \lambda K_F(\bar{v}, \alpha)$ with

$$\begin{aligned} L_F(\bar{v}, \alpha) &= (1-\bar{v})(1-F^n(q_t)) + \alpha\bar{v}nF^{n-1}(q_t)(1-F(q_t)) \qquad \text{(A.70)} \\ K_F(\bar{v}, \alpha) &= \int_{q_t}^{w_F} [\bar{v}(1-\alpha)A_t(q) + \bar{v}\alpha B_t(q)]dF^n(q) \end{aligned}$$

The following result implies that transferring money from the second highest prize to the highest prize would push the expected maximum performance further beyond the threshold.

Lemma A.16 *$K_F(\bar{v}, \alpha)$ is decreasing in α over* $[0, 1/2]$.

Proof. Notice that

$$K_F(\bar{v}, \alpha) = \int_{q_t}^{w_F} [\bar{v}A_t(q) + \alpha\bar{v}(B_t(q) - A_t(q))]dF^n(q)$$

so

$$\begin{aligned} \frac{\partial K_F(\bar{v}, \alpha)}{\partial \alpha} &= -[\bar{v}A_t(q_t) + \alpha\bar{v}(B_t(q_t) - A_t(q_t))](F^n)'(q_t)\frac{dq_t}{d\alpha} \\ &\quad +\bar{v}\int_{q_t}^{w_F} (B_t(q) - A_t(q))dF^n(q) \\ &= \bar{v}\int_{q_t}^{w_F} (B_t(q) - A_t(q))dF^n(q) \end{aligned}$$

where the last equality is from $A_t(q_t) = B_t(q_t) = 0$. Thus, it is sufficient to show

$$\int_{q_t}^{w_F} (B_t(q) - A_t(q))dF^n(q) < 0$$

The remainder of the proof has five steps.

Step 1. For any $q \in (q_t, w_F)$, $A_t(q) > 0$ and $A_t'(q) > 0$. This is obvious from the definition of $A_t(q)$.

Step 2. For q^* such that $F(q^*) = \frac{n-2}{n-1}$, we have $B_t'(q) > 0$ for $q \in (q_t, q^*)$ and $B_t'(q) < 0$ for $q \in (q^*, w_F)$. Since $B_t'(q) = (n-1)qF^{n-3}(q)f(q)((n-2) - (n-1)F(q))$, the property is true.

Step 3. $B_t(w_F) < 0$. From its definition, we have

$$B_t(w_F) = (n-1)\int_{q_t}^{w_F} sF^{n-3}(s)f(s)((n-2) - (n-1)F(s))ds$$

If we drop s in the integrand of $B_t(w_F)$, we obtain a similar integral

$$\begin{aligned} & (n-1)\int_{q_t}^{w_F} F^{n-3}(s)f(s)((n-2)-(n-1)F(s))ds \\ = \;& (n-1)(F^{n-1}(q_t)-F^{n-2}(q_t))<0 \end{aligned}$$

Notice that the integrand $F^{n-3}(s)f(s)((n-2)-(n-1)F(s))$ is positive over small values of s and negative over larger values of s. Then, if we multiply the integrand by the increasing function $h(s) = s$, the negative part of the integrand decreases more than the positive part increases, so $\int_{q_t}^{w_F} F^{n-3}(s)f(s)((n-2)-(n-1)F(s))ds < 0$ implies $\int_{q_t}^{w_F} sF^{n-3}(s)f(s)((n-2)-(n-1)F(s))ds < 0$.[32] Thus, $B_t(w_F) < 0$.

Step 4. there is at most one $q^{**} > q_t$ such that $A_t(q) > B_t(q)$ for $q > q^{**}$ and $A_t(q) < B_t(q)$ for $q < q^{**}$. If $B_t'(q_t) \leq A_t'(q_t)$, then $B_t'(q) - A_t'(q) = (n-1)qF^{n-3}(q)f(q)[(n-2)-nF(q)] < 0$ for $q > q_t$, so $B_t'(q) < A_t'(q)$ for $q > q_t$ and q^{**} does not exist. If $B_t'(q_t) > A_t'(q_t)$, then $B_t(q)-A_t(q)$ is positive for q slightly above q_t. Recall that $B_t(w_F) < 0 < A_t(w_F)$, so there is $q^{**} > q_t$ such that $A_t(q^{**}) = B_t(q^{**})$. Since $B_t'(q) - A_t'(q) = (n-1)qF^{n-3}(q)f(q)[(n-2)-nF(q)]$ can be zero at most once, $B_t(q) - A_t(q)$ cannot cross zero multiple times.

Step 5. $\int_{q_t}^{w_F}(B_t(q)-A_t(q))dF^n(q) < 0$. From Step 4, if there is no $q^{**} > q_t$ such that $A_t(q^{**}) = B_t(q^{**})$, then $B_t(q) < A_t(q)$ for $q > q_t$. Thus, $\int_{q_t}^{w_F}(B_t(q)-A_t(q))dF^n(q) < 0$. Suppose there is exactly one $q^{**} > q_t$ such that $A_t(q^{**}) = B_t(q^{**})$. Then, we have

$$\begin{aligned} & \int_{q_t}^{w_F}(B_t(q)-A_t(q))dF^n(q) \\ = \;& (n-1)\int_{q_t}^{w_F}\int_{q_t}^{q} sF^{n-3}(s)[(n-2)-nF(s)]dF(s)dF^n(q) \end{aligned}$$

[32] Formally, consider $L(s) < 0$ for $s \in (a,b)$ and $L(s) > 0$ for $s \in (b,c)$. Then, $\int_a^c L(s)ds > 0 \Rightarrow \int_a^c h(s)L(s)ds > 0$ for $h(s) > 0$ and $h'(s) > 0$.

If we drop s in the integrand, we obtain a similar integral:

$$
\begin{aligned}
& (n-1)\int_{w_F}^{w_F}\int_{q_t}^{q} F^{n-3}(s)[(n-2)-nF(s)]dF(s)dF^n(q) \\
= & (n-1)\int_{F(q_t)}^{1}\int_{F(q_t)}^{k} [z^{n-3}((n-2)-(n-1)z)-z^{n-2}]dzdk \\
= & (n-1)\Big[\frac{n}{2(n-1)}-\frac{n^2}{(n-1)(2n-1)}-F^{n-2}(q_t)+\frac{n}{n-1}F^{n-1}(q_t) \\
& +F^{2n-2}(q_t)\frac{n-2}{2(n-1)}-F^{2n-1}(q_t)\frac{n}{2n-1}\Big] \\
= & (n-1)\left[\underbrace{F^{n-2}(1-F)\frac{(n+1)F^n-2n+1}{2n-1}}_{\leq 0}-\underbrace{\frac{n}{n-1}\frac{(1-F^{2n-2})+F^{2n-2}(1-F)}{2(2n-1)}}_{>0}\right] \\
< & 0
\end{aligned}
$$

where the first equation is from changing variables $F(s)=z$ and $F(q)=k$, and we omit the argument in $F(q_t)$ in the last equation.

Similar to Step 3, since the integrand $F^{n-3}(s)[(n-2)-nF(s)]$ is positive over smaller s and negative over larger s, when we multiply this integrand by the increasing function $h(s)=s$, the negative part of the integrand decreases more than the positive part increases. Thus, $(n-1)\int_{w_F}^{w_F}\int_{q_t}^{q}F^{n-3}(s)[(n-2)-nF(s)]dF(s)dF^n(q)<0$ implies $(n-1)\int_{w_F}^{w_F}\int_{q_t}^{q}sF^{n-3}(s)[(n-2)-nF(s)]dF(s)dF^n(q)<0$.[33] Therefore, $\int_{q_t}^{w_F}(B_t(q)-A_t(q))dF^n(q)<0$. ■

Lemma A.17 *Suppose λ is sufficiently small, then given the optimal purse, two equal prizes are optimal.*

Proof. Since $v^1=\bar{v}(1-\alpha)$ and $v^2=\bar{v}\alpha$. We can rewrite (A.66) as

$$q_tF^{n-1}(q_t)+\alpha q_tF^{n-2}(q_t)[(n-1)-nF(q_t)]=\frac{t}{\bar{v}} \tag{A.71}$$

If $\lambda=0$, the seeker's expected profit is $L_F(\bar{v},\alpha)$. Given $\bar{v}$, to maximize the expected profit, the optimal α solves

$$\max_{\alpha\in[0,0.5]}(1-\bar{v})(1-F^n(q_t))+\alpha\bar{v}nF^{n-1}(q_t)(1-F(q_t))$$

subject to (A.71).

[33]Formally, consider $L(t)<0$ for $t\in(a,b)$ and $L(t)>0$ for $s\in(b,c)$. Then, $\int_a^c\int_s^c L(t)dtds>0\Rightarrow\int_a^c\int_s^c h(t)L(t)dtds>0$ for any $h(t)>0$ and $h'(t)>0$.

We introduce some notations. For a challenge with idea quality distribution F, let $\bar{v}_F$ be the optimal purse and α_F be the optimal prize allocation. Additionally,

$$\begin{aligned} H(F,\alpha) &= -(1-F^n)+\alpha n F^{n-1}(1-F)<0 && \text{(A.72)} \\ \frac{\partial H(F,\alpha)}{\partial F} &= nF^{n-1}(1-2\alpha)+\alpha n F^{n-2}((n-1)-(n-2)F)>0 && \text{(A.73)} \\ \frac{\partial^2 H(F,\alpha)}{\partial F^2} &= n(n-1)(1-2\alpha)F^{n-2}+(n-1)(n-2)\alpha F^{n-3}(1-F)>0 && \text{(A.74)} \end{aligned}$$

Then, we can rewrite (A.70) as

$$L_F(\bar{v},\alpha) \quad = \quad 1-F^n(q_t)+\bar{v}H(F(q_t),\alpha) \tag{A.75}$$

and (A.71) as

$$\frac{q_t}{n}\frac{\partial H(F(q_t),\alpha)}{\partial F}=\frac{t}{\bar{v}}$$

For any given $\alpha \in [0,1/2]$, let $\bar{V}(\alpha)$ be the purse that maximizes $L_F(\bar{v},\alpha)$. We want to show $\frac{dL_F(\bar{V}(\alpha),\alpha)}{d\alpha}>0$. Namely, as α increases, the seeker's profit $L_F(\bar{V}(\alpha),\alpha)$ increases.

Consider the following maximization problem:

$$\begin{aligned} &\max_{V,q} 1-F^n(q)+\bar{v}H(F(q),\alpha) \\ \text{s.t.} \quad &\frac{q}{n}\frac{\partial H(F(q),\alpha)}{\partial F}=\frac{t}{\bar{v}} \end{aligned}$$

The maximum should be the same as $L_F(\bar{V}(\alpha),\alpha)$. Let the Lagrangian of the above problem be

$$\mathcal{L}(\bar{v},q,\mu,\alpha)=1-F^n(q)+\bar{v}H(F(q),\alpha)+\mu\left(\bar{v}\frac{q}{n}\frac{\partial H(F(q),\alpha)}{\partial F}-t\right)$$

Let V^*, q^*, μ^* be the maximizers of the Lagrangian and they satisfy the FOC

$$\frac{\partial \mathcal{L}(\bar{v},q,\mu,\alpha)}{\partial \bar{v}}=H(F(q^*),\alpha)+\mu^*\frac{q^*}{n}\frac{\partial H(F(q^*),\alpha)}{\partial F}=0 \tag{A.76}$$

By the Envelope Theorem, we have

$$\begin{aligned} \frac{dL_F(\bar{V}(\alpha),\alpha)}{d\alpha} &= \frac{\partial \mathcal{L}(\bar{v}^*,q^*,\mu^*,\alpha)}{\partial \alpha} \\ &= \bar{v}^* n F^{n-1}(q^*)(1-F(q^*))+\mu^*\bar{v}^*q^*F^{n-2}(q^*)[(n-1)-nF(q^*)] \end{aligned}$$

Solving for μ^* from (A.76) and substituting into the above equation, we obtain

$$\frac{dL_F(\bar{V}(\alpha),\alpha)}{d\alpha} \propto nF(q^*)(1-F(q^*)) + \frac{-H(F(q^*),\alpha)}{\frac{q^*}{n}\frac{\partial H(F(q^*),\alpha)}{\partial F}} q^*[(n-1)-nF(q^*)]$$

Recall that $\frac{\partial H(F(q^*),\alpha)}{\partial F} > 0$, so to show $\frac{dL_F(\bar{V}(\alpha),\alpha)}{d\alpha} > 0$, it is sufficient to show

$$nF(q^*)(1-F(q^*))\frac{q^*}{n}\frac{\partial H(F(q^*),\alpha)}{\partial F} - H(F(q^*),\alpha)q^*[(n-1)-nF(q^*)] > 0$$

Substituting (A.72) and (A.73) into the above inequality, we can rewrite it as

$$nF^n(q^*)(1-F(q^*)) - (1-F^n(q^*))[(n-1)-nF(q^*)] > 0$$

Collecting terms with n, we can rewrite the left hand side as

$$n(1-F(q^*)) - (1-F^n(q^*)) = (1-F(q^*))[n-(1+F(q^*)+...+F^{n-1}(q^*))] > 0$$

Notice that if $F(q^*)=1$, no solvers submit solutions, so $F(q^*)<1$ for any $\alpha \in [0,1/2]$ and the last inequality is strict. Thus, $\frac{dL_F(\bar{V}(\alpha),\alpha)}{d\alpha} > 0$ for any $\alpha \in [0,1/2]$. This means if $\lambda = 0$, the seeker's optimal choice $(\bar{v},\alpha)$ must satisfy $\alpha = 1/2$.

Suppose λ is sufficiently small but positive. Then, the seeker's expected profit is $\Pi_F(\bar{v},\alpha) = L_F(\bar{v},\alpha) + \lambda K_F(\bar{v},\alpha)$. Let $\bar{V}(\alpha)$ be the optimal purse size to maximize $\Pi_F(\bar{v},\alpha)$ for a given α. Since λ is sufficiently small, by continuity we have $\frac{d\Pi_F(\bar{V}(\alpha),\alpha)}{d\alpha} > 0$. Thus, equal prizes are optimal for λ sufficiently small. ■

Before moving on to the next lemma, we introduce some new notation. Let q_F be the random variable with c.d.f. F and q_G be the random variable with c.d.f. G. Moreover, let $x_F = q_F F^n(q_F)$ with c.d.f. $\hat{F}$ and $x_G = q_G G^{n-1}(q_G)$ with c.d.f. $\hat{G}$. Define another two random variable as

$$\psi_F(q_F) = q_F F^{n-1}(q_F) + \alpha q_F F^{n-2}(q_F)[(n-1)-nF(q_F)]$$
$$\psi_G(q_G) = q_G G^{n-1}(q_G) + \alpha q_G G^{n-2}(q_G)[(n-1)-nG(q_G)]$$

and $\psi_F(q_F) \prec_{HR} \psi_G(q_G)$ means $\psi_G(q_G)$ dominates $\psi_F(q_F)$ in terms of the hazard rate.

Lemma A.18 *Suppose i) $\hat{G}$ or $\hat{F}$ has increasing hazard rate and ii) $\hat{G}$ or $\hat{F}$ is log-concave. Then, for any $\alpha \in [0,1/2]$, $x_F \prec_{LR} x_G \Rightarrow \psi_F(q_F) \prec_{HR} \psi_G(q_G)$.*

Proof. We prove in three steps.

Step I. We define a mapping from x_F to $\psi_F(q_F)$. By its definition, we have

$$\begin{aligned}\psi_F(q_F) &= q_F F^{n-1}(q_F) + \alpha q_F F^{n-1}(q_F)\left(\frac{n-1}{F(q_F)} - n\right) \\ &= x_F + \alpha x_F \left(\frac{n-1}{\hat{F}(x_F)} - n\right) \\ &\equiv \gamma_F(x_F)\end{aligned}$$

where the second equality is from $F(q_F) = \hat{F}(x_F)$. Similarly,

$$\psi_G(q_G) = x_G + \alpha x_G \left(\frac{n-1}{\hat{G}(x_G)} - n\right) \equiv \gamma_G(x_G)$$

Since x_F is increasing in q_F and $\psi_F(q_F)$ is also increasing in q_F, $\gamma_F(x_F)$ must be increasing in x_F. Similarly, $\gamma_G(x_G)$ is increasing in x_G.

Step II. We derive several properties of γ_F and γ_G: i) $\gamma_F(x) \leq \gamma_G(x)$ for $x \in (0, w_F)$, ii) $\gamma_G^{-1}(z) \leq \gamma_F^{-1}(z)$ for $z \in (0, \alpha w_F)$; iii) $\hat{F}(\gamma_F^{-1}(z)) \geq \hat{G}(\gamma_G^{-1}(z))$ for $z \in (0, \alpha w_F)$; iv) $\frac{\gamma_F^{-1}(z)}{\hat{F}(\gamma_F^{-1}(z))} \geq \frac{\gamma_G^{-1}(z)}{\hat{G}(\gamma_G^{-1}(z))}$ for $z \in (0, \alpha w_F)$; v) $\frac{\hat{g}(\gamma_G^{-1}(z))}{\hat{G}(\gamma_G^{-1}(z))} \geq \frac{\hat{f}(\gamma_F^{-1}(z))}{\hat{F}(\gamma_F^{-1}(z))}$ for $z \in (0, \alpha w_F)$; vi) $\frac{\hat{g}(\gamma_G^{-1}(z))}{1-\hat{G}(\gamma_G^{-1}(z))} \geq \frac{\hat{f}(\gamma_F^{-1}(z))}{1-\hat{F}(\gamma_F^{-1}(z))}$ for $z \in (0, \alpha w_F)$.

To see i), recall that $\hat{F} \prec_{LR} \hat{G}$ implies $\hat{F} \prec_{FOSD} \hat{G}$, so $\hat{F}(x) \geq \hat{G}(x)$. Then, the definitions of γ_F and γ_G imply $\gamma_F(x) \leq \gamma_G(x)$.

Property ii) is implied by i). For simpler notation, define $\gamma_G^{-1}(z) = \hat{x}_G$ and $\gamma_F^{-1}(z) = \hat{x}_F$. Thus, ii) can be rewritten as $\hat{x}_G \leq \hat{x}_F$.

To see iii), notice that the definitions of $\hat{x}_G$ and $\hat{x}_F$ imply

$$\hat{x}_F\left(1 - \alpha n + \alpha(n-1)\frac{1}{\hat{F}(\hat{x}_F)}\right) = \hat{x}_G\left(1 - \alpha n + \alpha(n-1)\frac{1}{\hat{G}(\hat{x}_G)}\right) = z \tag{A.77}$$

Since both $1 - \alpha n + \alpha(n-1)\frac{1}{\hat{F}(\hat{x}_F)} > 0$ and $1 - \alpha n + \alpha(n-1)\frac{1}{\hat{G}(\hat{x}_G)} > 0$, property ii) implies $\hat{F}(\hat{x}_F) \geq \hat{G}(\hat{x}_G)$.

For iv), we can first rewrite (A.77) as

$$\alpha(n-1)\frac{\hat{x}_F}{\hat{F}(\hat{x}_F)} - (\alpha n - 1)\,\hat{x}_F = \alpha(n-1)\frac{\hat{x}_G}{\hat{G}(\hat{x}_G)} - (\alpha n - 1)\,\hat{x}_G$$

Then, $\hat{x}_F \geq \hat{x}_G$ implies $\frac{\hat{x}_F}{\hat{F}(\hat{x}_F)} \geq \frac{\hat{x}_G}{\hat{G}(\hat{x}_G)}$.

To show v), notice that

$$\frac{\hat{g}(\hat{x}_G)}{\hat{G}(\hat{x}_G)} \geq \frac{\hat{g}(\hat{x}_F)}{\hat{G}(\hat{x}_F)} \geq \frac{\hat{f}(\hat{x}_F)}{\hat{F}(\hat{x}_F)}$$

where the first inequality is because $\frac{\hat{g}(x)}{\hat{G}(x)}$ decreases in x from the logconcavity assumption, and the second inequality is from the reverse hazard rate dominance implied by $x_F \prec_{LR} x_G$. While we prove under the assumption that $\hat{G}$ is log-concave, the proof is the same if $\hat{F}$ is log-concave.

Property vi) follows from

$$\frac{\hat{g}(\hat{x}_G)}{1-\hat{G}(\hat{x}_G)} \geq \frac{\hat{g}(\hat{x}_F)}{1-\hat{G}(\hat{x}_F)} \geq \frac{\hat{f}(\hat{x}_F)}{1-\hat{F}(\hat{x}_F)}$$

where the first inequality is from the increasing hazard rate assumption that $\frac{\hat{g}(x)}{1-\hat{G}(x)}$ increasing in x, and the second is from the hazard rate dominance implied from $x_F \prec_{LR} x_G$. While we prove under the assumption that $\frac{\hat{g}(x)}{1-\hat{G}(x)}$ increases in x, the proof is the same if $\frac{\hat{f}(x)}{1-\hat{F}(x)}$ increases in x.

Step III. We prove $\psi_F(q_F) \prec_{HR} \psi_G(q_G)$. Recall that $\psi_F(q_F) = \gamma_F(x_F)$, so $\psi_F(q_F)$ has a c.d.f. $\hat{F}(\gamma_F^{-1}(z))$. Similarly, $\psi_G(q_G)$ has a c.d.f. $\hat{G}(\gamma_G^{-1}(z))$. Therefore, for $\psi_F(q_F) \prec_{HR} \psi_G(q_G)$, we need to show $\frac{1-\hat{G}(\gamma_G^{-1}(z))}{1-\hat{F}(\gamma_F^{-1}(z))}$ increases in z, or equivalently,

$$\frac{-\hat{g}(\hat{x}_G)}{1-\hat{G}(\hat{x}_G)} \frac{\hat{x}_G}{dz} > \frac{-\hat{f}(\hat{x}_F)}{1-\hat{f}(\hat{x}_F)} \frac{\hat{x}_F}{dz} \tag{A.78}$$

From (A.77), we have

$$\frac{d\hat{x}_G}{dz} = \frac{1}{1+\alpha\left(\frac{n-1}{\hat{G}(\hat{x}_G)} - n\right) - \alpha(n-1)\frac{\hat{x}_G}{\hat{G}(\hat{x}_G)}\frac{\hat{g}(\hat{x}_G)}{\hat{G}(\hat{x}_G)}}$$

Therefore,

$$
\begin{aligned}
\text{LHS of (A.78)} &= \frac{1+\alpha\left(\dfrac{n-1}{\hat{G}(\hat{x}_G)}-n\right)-\alpha(n-1)\dfrac{\hat{x}_G}{\hat{G}(\hat{x}_G)}\dfrac{\hat{g}(\hat{x}_G)}{\hat{G}(\hat{x}_G)}}{\dfrac{\hat{g}(\hat{x}_G)}{1-\hat{G}(\hat{x}_G)}} \\
&= \alpha\frac{1-\hat{G}(\hat{x}_G)}{\hat{g}(\hat{x}_G)}+\alpha(n-1)\frac{1-\hat{G}(\hat{x}_G)}{\hat{G}(\hat{x}_G)}\left(\frac{1-\hat{G}(\hat{x}_G)}{\hat{g}(\hat{x}_G)}-\frac{\hat{x}_G}{\hat{G}(\hat{x}_G)}\right) \\
&\geq \alpha\frac{1-\hat{F}(\hat{x}_F)}{\hat{f}(\hat{x}_F)}+\alpha(n-1)\frac{1-\hat{F}(\hat{x}_F)}{\hat{F}(\hat{x}_F)}\left(\frac{1-\hat{F}(\hat{x}_F)}{\hat{f}(\hat{x}_F)}-\frac{\hat{x}_F}{\hat{F}(\hat{x}_F)}\right) \\
&= \text{RHS of (A.78)}
\end{aligned}
$$

where the inequality is from $\frac{1-\hat{G}(\hat{x}_G)}{\hat{g}(\hat{x}_G)} \geq \frac{1-\hat{F}(\hat{x}_F)}{\hat{f}(\hat{x}_F)}$ due to vi), $\frac{1-\hat{G}(\hat{x}_G)}{\hat{G}(\hat{x}_G)} \geq \frac{1-\hat{F}(\hat{x}_F)}{\hat{F}(\hat{x}_F)}$ due to iii), and $\frac{\hat{x}_G}{\hat{G}(\hat{x}_G)} \leq \frac{\hat{x}_F}{\hat{F}(\hat{x}_F)}$ due to iv). ■

Lemma A.19 *Suppose i) $\hat{G}$ or $\hat{F}$ has increasing hazard rate, and ii) $\hat{G}$ or $\hat{F}$ is log-concave. Then, if $\lambda = 0$, the optimal purses satisfy $\bar{V}_F(\lambda) \geq \bar{V}_G(\lambda)$.*

Proof. From (A.75), we have $L_F(\bar{v},\alpha) = 1-\tilde{F}^n\left(\frac{t}{\bar{v}}\right)+\bar{v}H(\tilde{F}\left(\frac{t}{\bar{v}}\right),\alpha)$. Therefore,

$$
\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} = -\frac{t}{\bar{v}^2}\frac{d\left(1-\tilde{F}^n\left(\frac{t}{\bar{v}}\right)\right)}{d\left(\frac{t}{\bar{v}}\right)}+H\left(\tilde{F}\left(\frac{t}{\bar{v}}\right),\alpha\right)-\frac{t}{\bar{v}}\frac{dH\left(\tilde{F}\left(\frac{t}{\bar{v}}\right),\alpha\right)}{d\tilde{F}\left(\frac{t}{\bar{v}}\right)}\tilde{f}\left(\frac{t}{\bar{v}}\right) \tag{A.79}
$$

Letting $x = t/\bar{v}$, we can rewrite $\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ as

$$
\frac{1}{\bar{v}} < \frac{H(\tilde{F}(x),\alpha)-x\dfrac{dH(\tilde{F}(x),\alpha)}{d\tilde{F}(x)}\tilde{f}(x)}{-x(\tilde{F}^n)'(x)} \tag{A.80}
$$

Thus, for $\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ implying $\frac{\partial L_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$, it is sufficient to have

$$
\frac{H(\tilde{F},\alpha)-x\dfrac{dH(\tilde{F},\alpha)}{d\tilde{F}}\tilde{f}}{-(\tilde{F}^n)'} \leq \frac{H(\tilde{G},\alpha)-x\dfrac{dH(\tilde{G},\alpha)}{d\tilde{G}}\tilde{f}}{-(\tilde{G}^n)'} \tag{A.81}
$$

where the arguments of functions are omitted. Using (A.72) and (A.73), we can rewrite the above condition as

$$\frac{1-\tilde{F}^n}{(\tilde{F}^n)'} - \alpha\frac{1-\tilde{F}}{\tilde{f}} + \alpha x(n-1)\frac{1}{\tilde{F}} \leq \frac{1-\tilde{G}^n}{(\tilde{G}^n)'} - \alpha\frac{1-\tilde{G}}{\tilde{f}} + \alpha x(n-1)\frac{1}{\tilde{G}} \tag{A.82}$$

Notice that for $q \in [0, w_F]$, the range for $\psi_F(q)$ is $[0, w_F(1-\alpha)]$. Moreover, the range for $\psi_G(q)$ is $[0, w_G(1-\alpha)]$. Since $w_F \geq w_G$, if (A.82) holds for $x \in [0, w_F(1-\alpha)]$ and $\alpha = 1/2$, we have $\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ implies $\frac{\partial L_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$. Thus, given $\lambda = 0$ the optimal purses satisfy $\bar{V}_F(\lambda) \geq \bar{V}_G(\lambda)$.

In the remainder of the proof, we show (A.82) holds for $x \in [0, w_F(1-\alpha)]$. Recall that

$$\tilde{F}(z) = F(\psi_F^{-1}(z)) = \hat{F}(\gamma_F^{-1}(z)) = \hat{F}(\hat{x}_F)$$
$$\tilde{G}(z) = G(\psi_G^{-1}(z)) = \hat{G}(\gamma_G^{-1}(z)) = \hat{G}(\hat{x}_G)$$

In addition, $\hat{F}(\hat{x}_F) \geq \hat{G}(\hat{x}_G)$ due to iii) in the proof of Lemma A.18. Thus,

$$\tilde{F}(z) \geq \tilde{G}(z) \tag{A.83}$$

The first two terms of (A.82) are

$$\frac{1-\tilde{F}^n}{(\tilde{F}^n)'} - \alpha\frac{1-\tilde{F}}{\tilde{f}} = \left(\frac{1}{n}\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} - \alpha\right)\frac{1-\tilde{F}}{\tilde{f}} \tag{A.84}$$

Next, we verify that $\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n}$ decreases in $\tilde{F}$. Taking derivative w.r.t. $\tilde{F}$, we obtain

$$\begin{aligned}
\frac{d}{d\tilde{F}}\left(\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n}\right) &= \frac{\tilde{F}^{n-2}}{(\tilde{F}^{n-1}-\tilde{F}^n)^2}[-\tilde{F}^{2n} + n\tilde{F} - (n-1)] \\
&= -\frac{\tilde{F}^{n-2}}{(\tilde{F}^{n-1}-\tilde{F}^n)^2}(1-\tilde{F})[n - (1 + \tilde{F} + ... + \tilde{F}^{2n-1})] \\
&> 0
\end{aligned}$$

Thus, $\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n}$ decreases in $\tilde{F}$. In addition, we have

$$\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} \geq \lim_{\tilde{F}\to 1}\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} = \lim_{\tilde{F}\to 1}\frac{-n\tilde{F}^{n-1}}{(n-1)\tilde{F}^{n-2} - n\tilde{F}^{n-1}} = 1 \tag{A.85}$$

where the first equality is from L'Hôpital's rule.

Next, we show (A.82). We show above that $\tilde{F}(z) \geq \tilde{G}(z)$ and $\dfrac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n}$ decreases in $\tilde{F}$, so

$$0 < \frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} - \alpha \leq \frac{1-\tilde{G}^n}{\tilde{G}^{n-1}-\tilde{G}^n} - \alpha \tag{A.86}$$

where the first inequality is because $\dfrac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} > 1$. Recall that Lemma A.18 implies $\frac{1-\tilde{F}}{\tilde{f}} \leq \frac{1-\tilde{G}}{\tilde{g}}$, which combined with (A.86) implies

$$\left(\frac{1}{n}\frac{1-\tilde{F}^n}{\tilde{F}^{n-1}-\tilde{F}^n} - \alpha\right)\frac{1-\tilde{F}}{\tilde{f}} \leq \left(\frac{1}{n}\frac{1-\tilde{G}^n}{\tilde{G}^{n-1}-\tilde{G}^n} - \alpha\right)\frac{1-\tilde{G}}{\tilde{g}} \tag{A.87}$$

Therefore, (A.84) implies

$$\frac{1-\tilde{F}^n}{(\tilde{F}^n)'} - \alpha\frac{1-\tilde{F}}{\tilde{f}} \leq \frac{1-\tilde{G}^n}{(\tilde{G}^n)'} - \alpha\frac{1-\tilde{G}}{\tilde{g}}$$

which combined with (A.83) implies (A.82). Therefore, for $\lambda = 0$, we have $\bar{V}_F(\lambda) \geq \bar{V}_G(\lambda)$.

■

Lemma A.20 *Given F and G, there exist $\tilde{\lambda} > 0$ and $\tilde{\varepsilon} > 0$ such that*
1) $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ *and* $\frac{\partial \Pi_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$ *for* $\bar{v} > t/\tilde{\varepsilon}$ *and* $\lambda < \tilde{\lambda}$,
2) $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} \geq 0$ *for* $\bar{v} < \frac{t}{w_F(1-\alpha)-\tilde{\varepsilon}}$ *and* $\lambda < \tilde{\lambda}$.

Proof. According to (A.80), we have $\lim_{t/\bar{v}\to 0} \frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} = -1$. Therefore, $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ if $t/\bar{v}$ and λ are close to zero. Similarly, $\frac{\partial \Pi_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$ if $t/\bar{v}$ and λ are close to zero.

If $\bar{v} = \frac{t}{w_F(1-\alpha)}$, we have $q_F = w_F$ and $\tilde{F}(\frac{t}{\bar{v}}) = 1$. Thus, $\Pi_F(\frac{t}{w_F(1-\alpha)}, \alpha) = 0$ for any λ. According to (A.80), we have

$$\left.\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}}\right|_{\bar{v}=\frac{t}{w_F(1-\alpha)}} = n w_F(1-\alpha)^2 \tilde{f}\left(\frac{t}{\bar{v}}\right)\left[\frac{w_F}{t} - 1\right] \geq 0$$

where the last inequality is from $t < w_F$. Since $L_F(\bar{v},\alpha) = 0$ and $\frac{\partial L_F(\bar{v},\alpha)}{\partial \bar{v}} \geq 0$ at $\bar{v} = \frac{t}{w_F(1-\alpha)}$, we have $L_F(\bar{v},\alpha) \geq 0$ for $\bar{v}$ slightly above $\frac{t}{w_F(1-\alpha)}$. In addition, for $\bar{v}$ slightly above $\frac{t}{w_F(1-\alpha)}$, q_F is slightly below w_F, so $K_F(\bar{v},\alpha) > 0$ and $\Pi_F(\bar{v},\alpha) > 0$. Since $\Pi_F(\bar{v},\alpha) = 0$ at $\bar{v} = \frac{t}{w_F(1-\alpha)}$ and $\Pi_F(\bar{v},\alpha) > 0$ for slightly higher $\bar{v}$, we have $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} \geq 0$ for $\bar{v}$ slightly above $\frac{t}{w_F(1-\alpha)}$. Thus, there exist $\tilde{\lambda} > 0$ and $\tilde{\varepsilon} > 0$ such that 1) and 2) hold.

■

Proof of Proposition 6. For a sufficiently large λ, Lemma A.16 implies that winner-take-all is optimal. Therefore, Proposition 3 implies that $\bar{V}_G(\lambda) \geq \bar{V}_F(\lambda)$. Namely, scarcer ideas lead to a smaller budget for prizes.

For sufficiently small λ, Lemma A.17 shows that two equal prizes are optimal. Lemma A.19 shows $\bar{V}_G(\lambda) \leq \bar{V}_F(\lambda)$ for $\lambda = 0$. Next, we show $\bar{V}_G(\lambda) \leq \bar{V}_F(\lambda)$ for $\lambda < \tilde{\lambda}$.

If (A.82) holds with equality for some $x \in (0, w_F)$, then $\hat{F}(x) = \hat{G}(x)$ for all $x \in (0, w_F)$, and therefore $\bar{V}_G(\lambda) = \bar{V}_F(\lambda)$ for all λ. To see why, suppose otherwise that (A.82) holds with equality for $x \in (0, w_F)$ and $\hat{F}(y) > \hat{G}(y)$ for $y \in (0, w_F)$. Since (A.82) holds with equality for x, we must have $\hat{F}(x) = \hat{G}(x)$ and $\hat{f}(x) = \hat{g}(x) > 0$. Of course, this implies $x \neq y$. If $y > x$, recall that $\hat{F} \prec_{LR} \hat{G}$ requires $\frac{\hat{g}(x')}{\hat{f}(x')}$ to be non-decreasing for $x' \in (x, y)$. Therefore $\hat{f}(x) = \hat{g}(x)$ implies $\hat{g}(x') \geq \hat{f}(x')$ for $x' \in (x, y)$. Since $\hat{F}(x) = \hat{G}(x)$, we have $\hat{G}(y) \geq \hat{F}(y)$, which contradicts $\hat{F}(y) > \hat{G}(y)$. If $y < x$, then $\hat{F} \prec_{LR} \hat{G}$ implies $\hat{g}(x') \leq \hat{f}(x')$ for $x' \in (y, x)$. Recall that $\hat{G}(x) = \hat{F}(x)$, so $\hat{G}(y) \geq \hat{F}(y)$, which also contradicts $\hat{F}(y) > \hat{G}(y)$.

Next, we consider the case in which (A.82) holds with strict inequality for all $x \in (0, w_F)$. In this case, we show below that $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0 \Rightarrow \frac{\partial \Pi_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$ for $\lambda < \tilde{\lambda}$, which implies $\bar{V}_G(\lambda) \geq \bar{V}_F(\lambda)$ for $\lambda < \tilde{\lambda}$.

To see this, notice that if $\bar{v} > t/\tilde{\varepsilon}$, Lemma A.20 implies that $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ and $\frac{\partial \Pi_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$. If $\bar{v} < \frac{t}{w_F(1-\alpha)-\tilde{\varepsilon}}$, Lemma A.20 implies $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} \geq 0$. Thus, in order to show $\frac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0 \Rightarrow \frac{\partial \Pi_G(\bar{v},\alpha)}{\partial \bar{v}} < 0$, it is sufficient to consider the range $\bar{v} \in \left[\frac{t}{w_F(1-\alpha)-\tilde{\varepsilon}}, \frac{t}{\tilde{\varepsilon}}\right]$. Since (A.82) holds with strict inequality for all $x \in (0, w_F)$. there exists $\varepsilon' > 0$ such that the difference between the two sides of (A.82) is at least ε' for $\bar{v} \in \left[\frac{t}{w_F(1-\alpha)-\tilde{\varepsilon}}, \frac{t}{\tilde{\varepsilon}}\right]$. Recall that (A.82) is equivalent to (A.81), so

$$\frac{H(\tilde{F},\alpha) - x\dfrac{dH(\tilde{F},\alpha)}{d\tilde{F}}\tilde{f}}{-(\tilde{F}^n)'} + \varepsilon' \leq \frac{H(\tilde{G},\alpha) - x\dfrac{dH(\tilde{G},\alpha)}{d\tilde{G}}\tilde{f}}{-(\tilde{G}^n)'} \tag{A.88}$$

for $x \in [\tilde{\varepsilon}, w_F(1-\alpha) - \tilde{\varepsilon}]$, where $x = t/\bar{v}$. Moreover, we can rewrite $\dfrac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} < 0$ as

$$\frac{1}{\bar{v}} < \frac{H(\tilde{F}(x),\alpha) - x\dfrac{dH(\tilde{F}(x),\alpha)}{d\tilde{F}(x)}\tilde{f}(x) + \lambda\dfrac{\partial K_F(\bar{v},\alpha)}{\partial \bar{v}}}{-x(\tilde{F}^n)'(x)}$$

which is an analogue of (A.80). For $\bar{v} \in \left[\frac{t}{w_F(1-\alpha)-\tilde{\varepsilon}}, \frac{t}{\tilde{\varepsilon}}\right]$, in order to show $\dfrac{\partial \Pi_F(\bar{v},\alpha)}{\partial \bar{v}} <$

$0 \Rightarrow \dfrac{\partial \Pi_G(\bar{v}, \alpha)}{\partial \bar{v}} < 0$, it is sufficient to prove

$$\frac{H(\tilde{F}, \alpha) - x\dfrac{dH(\tilde{F}, \alpha)}{d\tilde{F}}\tilde{f} + \lambda\dfrac{\partial K_F(\bar{v}, \alpha)}{\partial \bar{v}}}{-(\tilde{F}^n)'} \leq \frac{H(\tilde{G}, \alpha) - x\dfrac{dH(\tilde{G}, \alpha)}{d\tilde{G}}\tilde{f} + \lambda\dfrac{\partial K_G(\bar{v}, \alpha)}{\partial \bar{v}}}{-(\tilde{G}^n)'}$$

for $x \in [\tilde{\varepsilon}, w_F(1-\alpha) - \tilde{\varepsilon}]$, which holds for small enough $\lambda > 0$ because of (A.88). ■